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^{*1} The convergence to the associated price through iterative price adjustments for given wages was first proved in Shiozawa, Y Chapter 2 Th.4.4.10 (p. 82) in Shiozawa, Y., Morioka, M., & Taniguchi, K. (2019). This section is indebted to that description. The book does not mention the convergence to the maximal wage face.

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^{*2} In this section and the following "Face Movement" section, we do not distinguish between C and \hat{C} for simplicity in the figures.

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概要

Traditionally, international value theory and theories of international trade have been formulated focusing on the boundary point of the "production possibility set" of goods. However, this approach has the drawback of considering the exterior, boundary, and interior of the production possibility set as qualitatively distinct entities. While the production possibility set is a projection onto the goods space of technologically feasible points satisfying labor constraints, such a narrow perspective results in the loss of valuable information and obscures important relationships.

There exists a structural relationship between the set of technologically feasible points within the entire commodity space (goods and labor) and the set within the value space (prices and wages). By leveraging this structural relationship, numerous features of international economics can be elucidated. In contrast, the relationship between the set of feasible production points in the goods space and the set of prices is insufficient to reveal these characteristics. Thus, shifting the focus from feasible production points in the goods space to technologically feasible points in the entire commodity space is crucial for international value theory.

This paper shifts the perspective from feasible production points in the goods space to technologically feasible points within the entire commodity space and analyzes this structural relationship. It derives various propositions arising from the economic characteristics of international "production of goods by goods."

This paper examines the proposition that "quantity adjustment with fixed prices is a universal adjustment mechanism in the international economy," enabling the analysis of unemployment under free trade. The study employs mathematical techniques, including linear algebra, the theory of convex polyhedral cones, and the theory of face lattices.

Furthermore, the concept of linkage is introduced to analyze the dependency of international wages. While similar to Graham's concept of link commodities, the concept of linkage presented in this paper is mathematically more refined and precise. Using this concept, the relationship between complete linkage among countries, the uniqueness of value, and the dimension of the technologically feasible set is analyzed, along with the degree of specialization.

Notation

All vectors are column vectors. Transposes of vectors are denoted by the transpose symbol. Vectors and scalar variables are represented in lowercase italics.

The x element of vector x is distinguished by subscript as x_i .

Subvectors excluding the i element of vector x are denoted by subscript $\setminus i$ as $x_{\setminus i}$.

Matrices are represented in uppercase bold italics. The size of matrices may be indicated by subscript. Matrix $\mathbf{X}_{N \times T}$ represents that matrix \mathbf{X} is a N -by- T matrix.

The columns of matrix \mathbf{X} are denoted with lower case letters with parentheses and subscripts, like $x_{(j)}$.

From the 2nd to the 6th columns of matrix \mathbf{X} , we use subscripts with capital letters, such as $\mathbf{X}_{(i:j)}$ (note that $x_{(j)}$ is a vector represented in lowercase, while $\mathbf{X}_{(i:j)}$ is a submatrix denoted in uppercase, following common conventions).

The 2nd row of matrix \mathbf{X} is denoted with lowercase letters and subscripts, like $x_{[i]}$.

From the 2nd to the 6th rows of matrix \mathbf{X} , we use subscripts with numerical values, following the notation of $\mathbf{X}_{[i:j]}$.

The (i, j) -th element of matrix \mathbf{X} is represented as either $x_{i,(j)}$ or x_{ij} (for consistency, the 2nd element of column vector $x_{(j)}$ is commonly represented as x_{ij} instead of $x_{(j)i}$).

The submatrix excluding the 2nd row of matrix \mathbf{X} is denoted with a subscript $\setminus i$ and expressed as $\mathbf{X}_{\setminus i}$.

The null matrix and zero vector are exceptional cases denoted in lowercase non-italic style. To distinguish multiple vectors using the same symbol, subscript Roman numerals are used as shown in x_I, x_{II} . x_1 denotes the first element of vector x , and x_I has a different meaning.

For two vectors, x and y , $x \geq y$ holds for all i , where $x_i \geq y_i$ and $x \geq y$ are $x_i \geq y_i$ and at least one i . $x_i > y_i$ and $x > y$ imply $x_i > y_i$ for all i . In scalar x and y , $x \geq y$ and $x > y$ are used without distinction, particularly when they have the same meaning.

Vectors using symbol $\mathbf{1}$ and \mathbf{e} carry special meanings. A column vector of dimension K with all elements being 1 is denoted by $\mathbf{1}_K$, where the subscript K indicating the dimension is added initially to distinguish from the scalar 1. The subscript may be omitted when the context is clear, and confusion with scalars is avoided.

A vector of dimension K where only the i element is 1 and the rest are 0 is represented by \mathbf{e}_i . To prevent confusion with the i element of vector \mathbf{e} , the symbol \mathbf{e} is used exclusively for this special vector unless otherwise stated.

The set of points in the K -dimensional Euclidean space E^K , where all elements are positive, is denoted by E_{++}^K and referred to as the *first quadrant*. The set denoted by E^K is a subset where all elements are non-negative, referred to as the "non-negative orthant."

The set P and Q are given. For two sets, $P \supset Q$ indicates that P includes Q , and $P \setminus Q$ implies

that $P \cap Q^c$ while $P \not\supseteq Q$ means $P \supset Q$ and $P \neq Q$.

1 Basic Variables and Assumptions

1.1 Basic Variables

The international economy consists of M countries, N types of goods, and is characterized by T technologies. For the sake of brevity, the sets of country numbers, goods numbers, and technology numbers are also denoted by M , N , and T .

The combination of goods and labor is called a "commodity." Each technology is represented by a pair consisting of a M -dimensional labor input vector l and N -dimensional input and output vectors for goods, denoted as a and b respectively. Production involves taking l and a as inputs and producing b as outputs, where $l \geq 0$, $a \geq 0$, $b \geq 0$.

Assuming that a single technology produces only one type of good (referred to as the single-product assumption), b is positive for the elements related to the production of goods and zero for others.

The net production vector of goods is denoted as:

$$g := b - a \quad (1.1)$$

When the technology produces the good numbered n ,

$$g_n := b_n - a_n, \quad (1.2)$$

$$g_i := -a_i, \quad i \neq n. \quad (1.3)$$

A vector c combining the net production vector with the labor input vector represents a single technology.

$$c := \begin{pmatrix} g \\ -l \end{pmatrix}. \quad (1.4)$$

The input elements of c and g are negative.

When c produces goods n , the n element of g is positive, while the other elements are either zero or negative. If c belongs to country m , the m element of l is positive, and the rest are zero.

There are T technologies, each distinguished by the subscript τ . The international economic technological situation is represented by a matrix, denoted as \mathbf{C} , consisting of T instances of $c_{(\tau)}$ lined up horizontally. Matrix \mathbf{C} is of type $(M + N) \times T$. Similarly, matrices denoted as \mathbf{A} , \mathbf{B} , \mathbf{L} , and \mathbf{G} , each consisting of T instances of $a_{(\tau)}$, $b_{(\tau)}$, $l_{(\tau)}$, $g_{(\tau)}$ lined up horizontally, represent different aspects. Here, the matrices \mathbf{C} , \mathbf{A} , \mathbf{B} , \mathbf{L} , and \mathbf{G} are respectively referred to as the technology matrix, input matrix, output matrix, labor input matrix, and net output matrix.

The goods produced by technology τ and the country to which the technology τ belongs are denoted as $\gamma(\tau)$ and $\chi(\tau)$, respectively. The goods represented by $\gamma(\tau)$ are referred to

as goods of technology τ or simply as products, while the country corresponding to $\chi(\tau)$ is identified as the country of affiliation of technology τ . Conversely, the technology τ is recognized as the technology of country $\chi(\tau)$.

The vector representing the labor endowment of each country is denoted as \hat{l} .

The n -dimensional vector p has its n -th element indicating the price of goods n . The ****price vector**** denoted by p or simply ****price****.

The m -th element of the ****wage vector****, w , represents the nominal wages of country m . The ****wage vector**** w is also referred to as ****wages****.

The pair of p and w denoted by $q' := [p' \ w']$ is called the ****value vector**** or simply ****value****.

To analyze an international economy with positive markup rates set for prices, a modified version of the technology matrix is defined known as the ****modified technology matrix****. The markup rate for goods $\gamma(\tau)$ produced by country $\chi(\tau)$ is denoted by α_τ . When prices are determined by the following equation, they are said to be set according to the ****markup principle****, known as ****pricing based on the markup principle****. Prices determined by the markup principle are set by the following equation. When we divide both sides of Equation by $(1 + \alpha_\tau)$, the equation simplifies to:

$$p_{\gamma(\tau)} \frac{b_{\gamma(\tau)}(\tau)}{1 + \alpha_\tau} = p' a_{(\tau)} + w_{\chi(\tau)} l_{\chi(\tau)}(\tau). \quad (1.5)$$

We define the modified output vector as:

$$\frac{b_{\gamma(\tau)}(\tau)}{1 + \alpha_\tau} \quad (1.6)$$

By redefining technology matrix \mathbf{C} with this modified output vector, the analysis can be simplified, particularly when the markup rate is zero. In the following, unless otherwise specified, variables are assumed to be defined in the modified technology matrix base. The original net output matrix is denoted by $\tilde{\mathbf{G}}$.

1.2 Technologically Feasible Set and Fundamental Assumptions

【定義 1 Technologically Feasible Point: The product of the technology matrix \mathbf{C} and the operating scale $x \geq 0$ is referred to as a technologically feasible point.

【定義 2 Technologically Feasible Set: The entirety of technologically feasible points is called the technologically feasible set.

The technologically feasible set is a convex polyhedral cone spanned by \mathbf{C} . This polyhedral cone can be represented by either (\mathbf{C}) or (C) . When expressed as (C) , C is the set of points when the column vector $c_{(\tau)}$, $\tau \in T$, is considered as a point in E^{M+N} . In some cases, (C) is written as the polyhedral cone C .

$$A \text{ polyhedral cone } C := (C) := (\mathbf{C}) := \{ y \in E^{M+N} \mid y = \mathbf{C}x, x \geq 0 \}. \quad (1.7)$$

【定義 3Technology set: The set of points (C) obtained by regarding the columns of the technology matrix E^{M+N} as points in C is called the technology set.

【定義 4Commodity component and labor component: A technologically feasible point y is divided into two subvectors y_G and y_L .

$$y' = (y'_G : y'_L), \text{ where } y_G \text{ is the commodity component and } y_L \text{ is the labor component.} \quad (1.8)$$

$$(1.9)$$

【定義 5Output Quantity, Input Quantity, Net Output Quantity: We refer to Bx as the output quantity, Ax as the input quantity, and $(B - A)x$ as the net output quantity. These are also commonly known as production, input, and net production.

【定義 6Technology is Productive: A technology is considered productive if there exists an operating scale x^* such that all goods have positive net production, i.e., when $y_G^* = Gx^* > 0$, $x^* \geq 0$, exists.

It should be noted that this definition uses the technology matrix adjusted by the markup rate.

【定義 7All Countries' Technologies are Weakly Productive: A country is said to have weakly productive technologies if it can produce at least one type of good through the operation of its domestically owned technologies. In other words, if for any given $m \in M$, $\chi(\tau) \neq m$ implies $x_\tau = 0$ for some $x \geq 0$ and $y_G = Gx \geq 0$, then all countries' technologies are considered to be weakly productive.

【仮定 1Basic Assumptions: We assume the following regarding technology:

1. Assumption of single production: For all $1 \leq \tau \leq T$, there is $b_{\gamma(\tau),(\tau)} = 1$, and for other $n \neq \gamma(\tau)$, $1 \leq n \leq N$, there is $b_{n,(\tau)} = 0$.
2. Assumption of labor indispensability: For all $1 \leq \tau \leq T$, there is $l_{\chi(\tau),(\tau)} > 0$.
3. Assumption of productive technology: Technologies satisfy the "productive" condition as defined in Definition 6.
4. Assumption of weakly productive technology for all countries: Technologies in all countries satisfy the "weakly productive" condition as defined in Definition 7.
5. Assumption of positive labor endowments: Labor endowments $\hat{l} = (\hat{l}_1 \dots \hat{l}_M)'$ in each country are positive. $\hat{l} > 0$.

Certainly! Here is the English translation of the given Japanese economic paper using professional and academic language:

4 (weakly productive) is a weaker assumption than the assumption that each country can produce all goods in net terms.

Due to the indispensability of labor (Assumption 2), the technologically feasible set represented by (C) does not contain a straight line. Therefore, the matrix representing (C) (referred to as the *frame matrix*) is unique, and (C) is spanned by an extreme half-line (M. Gerstenhaber Theorem 9 in Cowles Commission (1951) (p.305))^{*3}. From this point onwards, the technology matrix C is assumed to be uniquely represented by the frame matrix.

【仮定 2Rank of Technology Matrix: The rank of the technology matrix C is $M + N$.

Suppose the rank of C is less than $M + N$. For instance, assume it is $M + N - 1$. Then, there exists a non-zero vector q such that $q'C = 0$ holds. This implies that all technologies are orthogonal to vector q . Given that there is no reason to believe that each technology vector has a special relationship with one another, Assumption 2 is considered reasonable.

From Assumption 2, the number of columns T of matrix C is at least $M + N$, but generally there are no other constraints on T . Similarly, the rank of the frame matrix of the technology face in r dimensions is r , but there are no constraints on the number of columns other than being at least r .

The vectors of matrix C are extreme points, so they cannot be expressed as non-negative linear combinations of other vectors, but they may be expressible as linear combinations without the non-negativity constraint.

2 Inequality Representation, Face, Face Lattice of Technologically Feasible Set

2.1 Inequality Representation, Face, Face Lattice of Technologically Feasible Set

Hereafter, the term polyhedral cone C is frequently used.

A polyhedral cone can be represented as the solution set of a system of homogeneous inequalities (Weyl's Theorem). For a proof, refer to Stoer and Witzgall (2012) p. 56. Using Weyl's theorem, the technologically feasible set (C) equals the solution set of homogenous linear inequalities

$$(C) = \{y \in E^{M+N} \mid y'Q \leq 0\} = \{y \in E^{M+N} \mid y'q_{(j)} \leq 0, 1 \leq j \leq K\} \quad (2.1)$$

where Q is a non-redundant matrix of type $(M + N) \times K$, meaning that removing any column

^{*3} A convex polyhedral cone that does not contain a straight line is called *pointed*, and such a cone is referred to as a *sharp convex polyhedral cone*.

of Q alters the set in (2.1).

【定義 8Singular Inequality: An inequality from (2.1) that holds with equality for all points in (C) is referred to as a singular inequality.

The presence of a singular inequality results in the polyhedral cone being contained in a hyperplane defined by the singular inequality, leading to the technology matrix, C , having a rank less than $M+N$, contradicting Assumption 2. Hence, Assumption 2 implies the absence of singular inequalities.

【定義 9Boundary Hyperplane: A *boundary hyperplane* of a polyhedral cone C is a hyperplane orthogonal to the column vector Q and passing through the vertex $q_{(j)}$, $1 \leq j \leq K$, of the cone.

【定義 10Face and Proper Face: A *face* of a polyhedral cone C is an intersection of one or more boundary hyperplanes with the cone C , the cone itself, the empty set \emptyset , etc. A face of C that is not the cone itself is called a *proper face*.

From the definition of a face, we have:

【命題 1Intersection of Faces: The intersection of two faces is a face.

【定義 11Frame Technology and Frame Matrix of Face F : Each technique that spans face F is referred to as the *frame technology* of face F . The matrix obtained by arranging all frame technologies of face F horizontally is called the *frame matrix* of face F .

The frame matrix of face F is a submatrix of the technology matrix C . The frame matrix of face F is sometimes referred to as the technology matrix of face F .

【定義 12Dimensional Space and Dimension of a Set: The minimum linear subspace containing set S is called the *dimensional space* (or *dimension space*) of set S , denoted by $D\{S\}$. The dimension of $D\{S\}$ is denoted by $d\{S\}$ and referred to as the dimension of set S .

【定義 13Boundary point and interior point: A point belonging to a proper face of a polyhedral cone C is referred to as a boundary point. Any point that is not a boundary point is called an interior point.

【定義 14An interior point, in terms of the relative topology of a polyhedral cone C , refers to interior points considered in the sense of topological space theory. In an interior point, strict inequalities, other than the singular inequality in (2.1), hold (Stoer and Witzgall, 2012, p. 38).

The set of all faces of the polyhedral cone C ,

$$\mathcal{F}(C) := \{F \neq \emptyset \mid F \text{ is a face of } C\}, \quad (2.2)$$

forms a finite number of complete face lattices defined by the order relation induced by the set inclusion relation (Stoer and Witzgall, 2012, p. 69).

—
【定義 15Cover: A face $F, G \in \mathcal{F}(C)$, $F \supsetneq G$ covers another face $F \supsetneq H \supsetneq G$ if there does not exist a face $H \in \mathcal{F}(C)$ such that F .

—
【定義 16Facet: When a face C covers a face $F \in \mathcal{F}(C)$, we refer to F as a facet of C .

A facet is a proper face of C with the maximum dimension.

【命題 2Dimension of Face and Rank of Frame Matrix: For a polyhedral cone with dimension C , the dimension of a face F and the rank of the frame matrix of face F (denoted as \mathbf{F}) are equal. This can be expressed as $d\{F\} = \text{rank}\{\mathbf{F}\}$.

Proof: The linear subspace spanned by \mathbf{F} is the same as $D\{F\}$. Since the dimension of the former is the rank of \mathbf{F} and the dimension of the latter is $d\{F\}$, we have established $d\{F\} = \text{rank}\{\mathbf{F}\}$. (End of proof)

【定義 17Base Matrix of Frame Matrix and Basic Technology: For a frame matrix of rank r , a submatrix consisting of r linearly independent column vectors is referred to as the *basis matrix* of the frame matrix \mathbf{F} , and these column vectors are called *basis vectors* or *basic technology*.

The choice of basis for a linear space is not unique, hence the choice of basis matrix is also not unique. Therefore, when referring to the basis vector (basic technology), the basis matrix containing it is determined.

Useful properties of the face lattice for this paper are summarized.

【命題 3Properties of the Face Lattice: For the face lattice $\mathcal{F}(C)$, the following statements hold with respect to $F, G \in \mathcal{F}(C)$.

1. If F covers G , then $d\{G\} = d\{F\} - 1$.
2. If $F \supset G$, then there exists a sequence of faces $F_i \in \mathcal{F}(C)$, $0 \leq i \leq n$, starting from $F_0 = F$ and ending with $F_n = G$, such that F_i covers F_{i+1} .
3. The face of a polyhedral cone C is the intersection of facets of C .

Proof: Refer to Theorem (2.12.3) and Theorem (2.13.9) on page 71 of Stoer and Witzgall (2012), as well as the proof of (2.13.7). (End of proof)

2.2 Polar Cone, Polar Mapping, Technology Face, Value Face

The polar cone of a polyhedral cone C , denoted as C^p , is the set of points with a non-positive inner product with all points in C .

【定義 18 Polar Cone or Pole: The set defined by the following equation within the technologically feasible set C is referred to as the polar cone or pole, denoted by C^p :

$$C^p = \{q \in E^{M+N} \mid \text{for all } y \in C, q'y \leq 0\} \quad (2.3)$$

$$= \{q \mid q'C \leq 0\} \quad (2.4)$$

The point q in C^p represents the value of the commodity vector. The first N elements of q are referred to as goods prices p and the remaining M elements as wages w . $q' = [p' \ w']$.

The reason for focusing on C^p is that the points in C^p become candidates for international value associated with the technologically feasible set C . Further elaboration will be provided in the following section .

The following properties of C^p are crucial.

【命題 4 Representation for a Polar:

$$C^p = (Q) \quad (2.5)$$

Proof: Refer to Stoer and Witzgall (2012) (2.8.4) (p.55). (End of Proof)

Equation (2.5) shows that the polar C^p is a polyhedral cone, defined by the inequality representation of polyhedral cone C by the matrix in Equation (2.1). Since C^p is also a polyhedral cone, it has a face lattice $\mathcal{F}(C^p)$, and Proposition 3 holds for this lattice.

【定義 19 Polar Mapping: The mapping \mathcal{P} from $\mathcal{F}(C)$ to $\mathcal{F}(C^p)$ defined by the following equation is referred to as the polar mapping.

$$\mathcal{P}(F) := F^\perp \cap C^p \quad (2.6)$$

$\mathcal{P}(F)$ is a face of C^p (Stoer and Witzgall, 2012, p. 70), and it possesses the following properties.

【命題 5 Properties of the polar mapping: The polar mapping \mathcal{P} defined by $\mathcal{P}(F) = F^\perp \cap C^p$ has the following properties:

1. involution,
2. an anti-isomorphism from the face lattice of the polyhedral cone C to the face lattice of the polar cone C^p , where an involution is achieved when a mapping inverts the order relation of $\mathcal{F}(C)$ and the structures of $\mathcal{F}(C)$ and $\mathcal{F}(C^p)$ are isomorphic,
3. $d\{F\} + d\{\mathcal{P}(F)\} = M + N$.

Proof: The proofs for 1 and 2 can be found in Theorem (2.13.2) on page 70 of Stoer and Witzgall (2012). The proof for 3 is as follows.[Title: Translation of Economic Paper from Japanese to English]

[Proposition 3]: Under , when F is a facet of C , then $d\{F\} = M + N - 1$, and \mathcal{P} are reverse isomorphic mappings, since C 's face F decreases in dimension by 1 each time, and thus \mathcal{P} is a reverse isomorphic mapping, then the dimension of $\mathcal{P}(F)$ increases by 1 each time. Hence, 3 is established. (End of proof)

To distinguish between the faces of technologically feasible set C and the extremum C^p , we assign names to each.

【定義 20 Technology Face and Value Face: The face of the technologically feasible set C is called the technology face, and the face of the extremum C^p is called the value face.

The reason for calling it the value face is that the positive subset of the value face becomes the value of international economy.

【定義 21 Value Face of Technology Face: $\mathcal{P}(F)$ is referred to as the value face (associated with) of technology

【定義 22 Price Face and Wage Face: The price component of the technology face $F \in \mathcal{F}(C)$ under the polar mapping $\mathcal{P}(F)$ is referred to as the price face associated with the technology face F , denoted by $\mathcal{P}_P(F)$.

$$\mathcal{P}_P(F) := \{p \mid q' = (p' \ w') \in \mathcal{P}(F)\} \quad (2.7)$$

【定義 23 Wage Face: The wage component of the polar mapping $\mathcal{P}(F)$ is called the wage face associated with the technology face F , denoted by $\mathcal{P}_W(F)$.

The inclusion relation of the price face and wage face inherits the inclusion relation of the value face.

【命題 6 Inclusion Relation of Price Face and Wage Face: If $\mathcal{P}(F) \supset \mathcal{P}(F')$, then $\mathcal{P}_P(F) \supset \mathcal{P}_P(F')$, and $\mathcal{P}_W(F) \supset \mathcal{P}_W(F')$.

Regarding the set operations of the face lattice of the technologically feasible set ($\mathcal{F}(C)$) and the face lattice of the poles ($\mathcal{F}(C^p)$), the following holds:

【命題 7 Value Face of the Intersection of Technology Face: $\mathcal{P}(F \cap G) = \mathcal{P}(F) + \mathcal{P}(G)$

Proof: If $C = \{y \mid y'Q \leq 0\}$, then $C^p = \{Qx \mid x \geq 0\}$ (by Proposition 4). Let's consider expressing the polar mapping of the technology face ($F \in \mathcal{F}(C)$) as a matrix in $\mathcal{P}(F)$. The face of $\mathcal{P}(F)$ corresponding to C^p can be expressed using a certain submatrix Q_F of Q .

Next, consider representing F^\perp in matrix form:

$$\begin{aligned} F^\perp &= F^\perp \cap (C^p \cup -C^p) \\ &= (F^\perp \cap C^p) \cup (F^\perp \cap -C^p) \\ &\quad \text{Since } F^\perp \text{ is a subspace, } F^\perp = -F^\perp \\ &= (F^\perp \cap C^p) \cup (-F^\perp \cap -C^p) \\ &= (F^\perp \cap C^p) \cup -(F^\perp \cap C^p) \end{aligned} \quad (2.8)$$

Given that $(F^\perp \cap C^p) = \{Q_F u \mid u \geq 0\}$, it follows that $-(F^\perp \cap C^p) = \{Q_F u \mid u \leq 0\}$.

Therefore,

$$F^\perp = \{Q_F u\}. \quad (2.9)$$

Finally, represent $\mathcal{P}(F \cap G) = (F \cap G)^\perp \cap C^p = (F^\perp + G^\perp) \cap C^p$ in matrix form^{*4}. Applying (2.9) to the technology face yields the set G , resulting in $G^\perp = \{Q_G v\}$. Representing the collection of Q_F , Q_G columns without duplication as a submatrix Q denoted by $Q_{F \cup G}$, we have

$$\begin{aligned} (F^\perp + G^\perp) \cap C^p &= \{Q_{F \cup G} w\} \cap \{Qx \mid x \geq 0\} \\ &= \{Q_{F \cup G} w \mid w \geq 0\} \\ &= \{Q_F u \mid u \geq 0\} + \{Q_G v \mid v \geq 0\} \end{aligned} \quad (2.10)$$

The final term is equivalent to $\mathcal{P}(F) + P(G)$. (End of proof)

The sum of the dimensions of the technology face and the associated value face is always equal to $M + N$ (Proposition 5, item 3). Therefore,

【命題 8】 Dimension of Value Face:

1. The value face associated with a r -dimension technology face is a $M + N - r$ -dimensional polyhedral cone.
2. The facet's value face is one-dimensional and uniquely determined except for scale.

【定義 24 Value Extreme Halfline and Wage Extreme Halfline: The one-dimensional value face of a facet is referred to as the value extreme halfline and the wage component of the value extreme halfline is referred to as the wage extreme halfline.

Any value face $\mathcal{P}(F)$ is spanned by a pair of the value extreme halfline. This is because, according to *Proposition 3*, as in , any technology face F is the intersection of a pair $F_1, \dots, F_l, \dots, F_L$ of facets. Therefore,

$$\mathcal{P}(F) = F^\perp \cap C^p = \left(\bigcap_{1 \leq l \leq L} F_l \right)^\perp \cap C^p = \left(\sum_{l=1}^L F_l^\perp \right) \cap C^p = \left(\sum_{l=1}^L (F_l^\perp \cap C^p) \right) \quad (2.11)$$

$$= \sum_{l=1}^L \mathcal{P}(F_l) \quad (2.12)$$

Hence, $\mathcal{P}(F)$ is a polyhedral cone spanned by $\mathcal{P}(F_l)$, and $\mathcal{P}(F_l)$ is a value extreme halfline.

Since any wage face $\mathcal{P}_W(F)$ is $\mathcal{P}_W(F) = \sum_{l=1}^L \mathcal{P}_W(F_l)$, it can be understood that it is spanned by a pair of wage extreme halflines. Therefore,

^{*4} Refer to page 103, equation (12) in Sawu (1974) for the second equality in $(F \cap G)^\perp = (F^\perp + G^\perp)$.

【命題 9 Value and wage surfaces spanned by extreme half-lines: Any value face is spanned by a pair of value extreme halflines, and any wage face is spanned by a pair of wage extreme halflines, forming a polyhedral cone.

By identifying a subset of $\mathcal{P}(F)$ that is orthogonal to F , we can find candidate "values" where the profit of C is non-negative and the profit of F is zero. Further restrictions on $\mathcal{P}(F)$ are needed to formally define the value.

3 Positive Region of Eligible Technology Face and Value Face, Price Face, Wage Face

The profits of all points evaluated at $\mathcal{P}(F)$ are zero for F , and the profits of points other than F are negative, displaying a favorable property for analyzing technological choices. However, in order for the points in $\mathcal{P}(F)$ to be economically meaningful, they must be positive, leading us to consider a subset of positive points in $\mathcal{P}(F)$.

【定義 25 Positive Region of Value Face, Price Face, and Wage Face: The positive subsets of the value face, price face, and wage face are respectively referred to as the positive region of the value face, the positive region of the price face, and the positive region of the wage face, or simply as the positive region of a value face, price face positive region, and wage face positive region.

Restricting the value of $\mathcal{P}(F)$ to the positive region, we define a mapping \mathcal{V} .

【定義 26 Value Positive Region Mapping, Price Positive Region Mapping, Wage Positive Region Mapping: The mapping from $\mathcal{F}(C)$ to C^p defined in (3.1) is called the value positive region mapping.

$$\mathcal{V}(F) := \{q \mid q \in \mathcal{P}(F), q > 0\} = \mathcal{P}(F) \cap E_{++}^{M+N} \quad (3.1)$$

The set of price components in $\mathcal{V}(F)$ is denoted by $\mathcal{V}_P(F)$. The mapping in \mathcal{V}_P is referred to as the *price positive orthant mapping*.

$$\mathcal{V}_P(F) = \{p = q_{1:N} \mid q \in \mathcal{V}(F)\} \quad (3.2)$$

The set of wage components in $\mathcal{V}(F)$ is represented by $\mathcal{V}_W(F)$. The mapping in \mathcal{V}_W is called the *wage positive orthant mapping*.

$$\mathcal{V}_W(F) = \{w = q_{N+1:M} \mid q \in \mathcal{V}(F)\} \quad (3.3)$$

$\mathcal{V}_P(F)$ and $\mathcal{V}_W(F)$ are the price positive orthant and wage positive orthant of F , respectively.

All points in $\mathcal{V}(F)$ are positive; hence, $\mathcal{V}(F)$ does not include the origin. The technology surface $\mathcal{V}(F) \neq \emptyset$ facing eligible technology surface $F \in \mathcal{F}(C)$ is referred to as the eligible technology face (eligible technology face), and the entirety of eligible technology face is represented by $\mathcal{E}(C)$.

【定義 27 Qualified Technical Surface and Its Whole:

$$F \in \mathcal{F}(C) \text{ is an eligible technology face } \stackrel{\text{defined as}}{\iff} \mathcal{P}(F) \cap E_{++}^{M+N} \neq \emptyset \quad (3.4)$$

$$\mathcal{E}(C) := \{F \mid F \text{ is an eligible technology face}\} \quad (3.5)$$

The term eligible technology face is sometimes simply referred to as eligible.

The eligible technology face $F \in \mathcal{E}(C)$ possesses the following properties:

1. Any point on $\mathcal{V}(F)$ is orthogonal to all points on F . Seen from the side of $\mathcal{V}(F)$, all points within F are equivalent.
2. On the other hand, it is not necessarily the case that $\mathcal{V}(F) = \mathcal{P}(F)$. In other words, from the perspective of the technology face, only a part of $\mathcal{P}(F)$ may be positive. From the perspective of F , not all points in $\mathcal{P}(F)$ are equivalent.

The technology face that is $\mathcal{V}(F) = \mathcal{P}(F)$ is called a fully eligible technology face (fully eligible face).

【定義 28 Fully Eligible Technology Face and Incompletely Eligible Technology Face: A technology face $F \in \mathcal{F}(C)$ is considered a fully eligible technology face if $\stackrel{\text{定義}}{\iff} \mathcal{P}(F) \setminus \{0\} \subset E_{++}^{M+N}$. An eligible technology face that is not fully qualified is referred to as an incompletely eligible technology face.

The value face of facet F is composed of only one half-line (within Proposition 8). If F is an eligible technology face, this half-line is contained in the strong positive quadrant excluding the origin. Since there is no other half-line in the value face, the entire value face is contained in the strong positive quadrant excluding the origin. In regard to facets, it is equivalent for a technology face to be an eligible technology face and to be a fully eligible technology face.

Within Proposition 3, as stated in , any technology face F is the intersection of a pair $F_1, \dots, F_l, \dots, F_L$ of facets. By (2.12) and Proposition 9, we have the following:

$$\begin{aligned} & \text{The set } \mathcal{P}(F) \setminus \{0\} \text{ belongs to the strong positive octant } E_{++}^{M+N}, \\ & \Leftrightarrow \text{The sets } \mathcal{P}(F_l) \setminus \{0\}, 1 \leq l \leq L, \text{ all belong to the strong positive octant } E_{++}^{M+N}, \\ & \Leftrightarrow \text{The faces } F_l, 1 \leq l \leq L, \text{ are all fully eligible technology faces.} \end{aligned} \quad (3.6)$$

and,

$$\begin{aligned}
& F \text{ is an eligible technology face,} \\
& \Leftrightarrow (\mathcal{P}(F) \setminus \{0\}) \cap E_{++}^{M+N} \neq \emptyset, \\
& \Leftrightarrow \text{At least one of the sets } \mathcal{P}(F_l) \setminus \{0\}, \text{ for } 1 \leq l \leq L, \text{ is contained in } E_{++}^{M+N}, \\
& \Leftrightarrow \text{At least one of the faces } F_l, \text{ for } 1 \leq l \leq L, \text{ is a fully eligible technology face.} \quad (3.7)
\end{aligned}$$

This concludes the proposition.

【命題 10 Properties of Eligible Technology Faces:

1. If a facet is an eligible technology face, then it is a fully eligible technology face.
2. A technology face F that is the intersection of facet $F_1, \dots, F_l, \dots, F_L$ is a fully eligible technology face if and only if all facets $F_l, 1 \leq l \leq L$, are fully eligible technology faces.
3. A technology face F that is the intersection of facet $F_1, \dots, F_l, \dots, F_L$ is an eligible technology face if and only if at least one facet $F_l, 1 \leq l \leq L$, is a fully eligible technology face.

【命題 11 Inclusion Relation of Eligible Technology Face, Fully Eligible Technology Face, and Positive Region of a Value Face: Let there be two distinct technology faces, denoted as F and G , with F containing G . The following statements hold:

1. $\mathcal{P}(F) \subset \mathcal{P}(G)$.
2. If F is an eligible technology face, then G is also an eligible technology face and $\mathcal{V}(F) \subset \mathcal{V}(G)$.
3. If G is a fully eligible technology face, then F is also a fully eligible technology face and $\mathcal{V}(F) \subset \mathcal{V}(G)$.

Proof: Proof of 1. $F \supset G \Rightarrow F^\perp \subset G^\perp \Rightarrow F^\perp \cap C^p \subset G^\perp \cap C^p$.

Proof of 2. F being an eligible technology face $\Leftrightarrow \mathcal{P}(F) \cap E_{++}^{M+N} \neq \emptyset$. Considering statement 1, we have $\mathcal{P}(G) \cap E_{++}^{M+N} \neq \emptyset$, and thus G is an eligible technology face. From $\mathcal{P}(F) \subset \mathcal{P}(G)$ to $\mathcal{P}(F) \cap E_{++}^{M+N} \subset \mathcal{P}(G) \cap E_{++}^{M+N}$, the condition $\mathcal{V}(F) \subset \mathcal{V}(G)$ holds.

Proof of 3. If G is a technologically feasible face, then it is fully productive $\Leftrightarrow \mathcal{P}(G) \subset E_{++}^{M+N} \Rightarrow \mathcal{P}(F) \subset E_{++}^{M+N} \Leftrightarrow F$ is a fully eligible technology face (End of proof).

Among the two properties of technology face, "productive" exhibits inheritance for higher-dimensional faces, while "eligibility" displays inheritance for lower-dimensional faces.

The eligibility of a facet is equivalent to its value half-line being positive, except at the origin. Therefore, to validate the eligibility of a facet, it suffices to examine its value half-line. As the

column vectors of Q within the inequality representation (2.1) of the technologically feasible set (C) define the value half-line, an eligible facet is orthogonal to positive column vectors^{*5}.

4 Proof of the Existence of Eligible Technology Face

The task at hand is to demonstrate the existence of a positive value in international economics, namely, to identify the sufficient conditions for the presence of an eligible technology face. Initially, we define the "maximal technologically feasible point" and elucidate its properties.

【定義 29Maximal Technologically Feasible Point: A technologically feasible point $y \in (C)$ is called a maximal technologically feasible point when there is no other technologically feasible point $\bar{y} \in (C)$ that satisfies $\bar{y} \geq y$.

【命題 12Existence of Maximal Technologically Feasible Point: If $y \in (C)$ is not a maximal technologically feasible point, then there exists a maximal technologically feasible point y^* such that $y^* \geq y$.

Proof: Let x denote the operating scale that achieves y , as represented by $l := Lx$.**Maximization Problem**

Consider the optimization problem of maximizing the sum of \hat{y} elements subject to the constraints that the quantity of goods is at least y and the labor usage is at most l .

Under the labor constraint $\hat{x}_\tau l_{\chi(\tau),(\tau)} \leq l_{\chi(\tau)}$, \hat{x}_τ has an upper limit of $\bar{x}_\tau := l_{\chi(\tau)}/l_{\chi(\tau)(\tau)}$. For any \hat{y} that satisfies the labor constraint, we have:

$$\hat{y}_n = g_{[n]} \hat{x} \leq \sum_{\{\tau | g_{n,(\tau)} > 0\}} g_{n\tau} \hat{x}_\tau \leq \sum_{\{\tau | g_{n,(\tau)} > 0\}} g_{n\tau} \bar{x}_\tau \quad (4.1)$$

Hence, \hat{y} is bounded above. Due to $\hat{y} \geq y$, \hat{y} is also bounded below. The set of \hat{y} satisfying the constraints is a closed set. Therefore, the set of \hat{y} satisfying the constraints is a compact set, and the maximization problem (4.1) has a solution, denoted by y^{*6} .

Next, let's demonstrate that y^* is a maximal technologically feasible point. Suppose, for contradiction, that y^* is not a maximal technologically feasible point, implying the existence of another \hat{y} , denoted as $\hat{y} \geq y^*$. \hat{y} satisfies the constraints (y labor input is at most l). However, this contradicts the fact that $\sum_{i=1}^{M+N} \hat{y}_i > \sum_{i=1}^{M+N} y^*_i$, resulting in the maximization of the objective function at y^* . Therefore, such a \hat{y} does not exist, indicating that y^* is indeed a maximal technologically feasible point. Assuming $y^* = y$, it contradicts the premise that y is not a maximal technologically feasible point, and thus, $y^* \geq y$ is true. (End of Proof)

^{*5} Refer to Hirai (1999) p.73, "5. Solution to Inequality Problems," for the method to derive Q from C using the CONVERT function.

^{*6} The y^* here is different from the one in Definition 6, stating that technology is productive.

Subsequently, we aim to demonstrate the existence of a maximal technologically feasible point where the net output quantity for all goods is positive. **【Definition 6】** states that if the net output quantity of operating scale x^* is maximal at the technologically feasible point, no further proof is required. Should y^* not be maximal at the technologically feasible point, Proposition 12 ensures the existence of a maximal technologically feasible point $y \geq y^*$. The commodity component of this maximal technologically feasible point is positive. Consequently,

【命題 13Positive Existence of Maximal Technologically Feasible Point: Assuming that within the framework of Assumption 1, item 3 holds, there exists a positive maximal technologically feasible point y for the net production of goods.

Since an interior point (as defined in Definition 13) allows movement in any direction, it cannot be a maximal technologically feasible point.

【命題 14Position of the Maximal Technologically Feasible Point: The maximal technologically feasible point is necessarily contained in the technology face.

【命題 15Necessary and Sufficient Conditions for the Maximal Technologically Feasible Point: The necessary and sufficient condition for y to be the maximal technologically feasible point is the existence of a positive vector that is orthogonal to y , such that

$$q' C \leq 0, q' y = 0, q > 0. \quad (4.2)$$

Proof: Refer to T. Koopmans Chapter III Theorem 4.3 (p. 61) in Cowles Commission (1951) and equation (4.7) on page 63 inside the proof.

【命題 16Sufficient Condition for the Existence of an Eligible Technology Face: If a technology face, denoted as F , has a maximal technologically feasible point y in its relative interior, then the value q orthogonal to y inside the technology face F intersects with all points of the technology face F . Since the polar mapping $\mathcal{P}(F)$ of F contains this value q , a technology face with a maximal technologically feasible point is considered an eligible technology face according to Proposition 15.

Proof: The proof that q intersects orthogonally with a relative interior point of the technology face F can be found in T. Koopmans Chapter III Lemma 4.6 on page 65 in Cowles Commission (1951). The orthogonality of q with a relative boundary point of the technology face F arises from the continuity of the inner product $q'y$.

【命題 17Necessary and Sufficient Conditions for an Eligible Technology Face: The necessary and sufficient condition for a technology face F to be considered an eligible technology face is that all technologically feasible points of F are maximal technologically feasible points.

Proof: If the technology face F is an eligible technology face, then according to Definition 27, there exists at least one positive value q . As per Proposition 16, this q is orthogonal to all points of F . Therefore, by Proposition 15, all points of F are maximal technologically feasible

points. Conversely, if all points of F are maximal technologically feasible points, then, according to Proposition 15, each point possesses positive value, hence F is an eligible technology face. (End of Proof)

Combining Propositions 13, 14, and 16, we obtain the following proposition.

【命題 18 Sufficient Condition for the Existence of an Eligible Technology Face: If assumption 1 with 3 (technology being productive) holds, then at least one eligible technology face exists.

Furthermore, a summary is provided regarding the benefits of technology evaluated in terms of values contained in the positive region of an eligible technology face, F .

【命題 19 Profit of Technology Evaluated at Positive Region of a Value Face $\mathcal{V}(F)$:

1. Since $\mathcal{V}(F) \subset \mathcal{P}(F) \subset C^p$, there is no technology generating positive profit.
2. Given $\mathcal{V}(F) \subset F^\perp$, the profit of frame technology in F is zero.
3. When evaluating a technology $C \setminus F$ other than technology F at a value q in the region of $\mathcal{V}(F)$ excluding the relative boundary of $\mathcal{P}(F)$, the profit of that technology is negative.

Proof: We only prove 3. Let the set of boundary hyperplanes defining technology face F be denoted by I . The value $q \in \mathcal{V}(F)$ in the region of $\mathcal{P}(F)$ excluding the relative boundary can be expressed as

$$q = \sum_{i \in I} \alpha_i q_{(i)} \quad (4.3)$$

where $q_{(i)}$ is a column vector of the matrix \mathbf{Q} representing the system of inequalities of the technologically feasible set. On the other hand, for any given technological ${}^*c_{(\tau)} \in C \setminus F^*$, at least one index within ${}^*I^*$ satisfies

$$q'_{(i)} c_{(\tau)} < 0. \quad (4.4)$$

The subset of ${}^*I^*$ that satisfies (4.4) is denoted by ${}^*J^*$.

$$q' c_{(\tau)} = \sum_{i \in I} \alpha_i q_{(i)} c_{(\tau)} = \sum_{i \in J} \alpha_i q_{(i)} c_{(\tau)} + \sum_{i \in I \setminus J} \alpha_i q_{(i)} c_{(\tau)} = \sum_{i \in J} \alpha_i q_{(i)} c_{(\tau)} < 0.. \quad (4.5)$$

(End of proof)

The third property is crucial. If the economy is situated within the relative interior of the eligible technology ${}^*F^*$, there is no rationale for adopting a frame matrix other than ${}^*F^*$. When the demand changes, the economy moves along the same value q and frame matrix \mathbf{F} while facing the open technology face $)F($ ^{*7}; hence, there is no change in value. Value changes due to a different cause, as explained in Section 8.3. This is also an important property in applications,

^{*7} $)F(:= \{\mathbf{F}x, x > 0\}$

used in the proofs of "Gains from Trade" in Section 12 and "Technological Change and Its Impact" in Section 13.

5 Mapping That Groups Technologies by Goods

5.1 Definition of Mapping

We explain an important mapping frequently used from this point onward. Consider the submatrices of the technology face F (including the technologically feasible set \mathbf{C} itself) corresponding to the input matrix \mathbf{A} , output matrix \mathbf{B} , and labor input matrix \mathbf{L} . All three submatrices are composed of columns from the original matrix. For the sake of notational brevity in this chapter, we denote the submatrices with the same symbols as the original matrices. We represent the number of columns (number of technologies included in the submatrix) of the submatrix by T' . The operating scale vector of the submatrix is also denoted by the same symbol x ($x \in E^{T'}$). The technology vector is standardized so that the positive elements of the output matrix \mathbf{B} become 1. We define the mapping $\phi : E^{T'} \rightarrow E^N$ that transfers the operating scale x to the output quantity for each type of goods as follows:

$$\phi(x)_n = \sum_{\gamma(\tau)=n} x_\tau, \quad 1 \leq n \leq N. \quad (5.1)$$

The mapping $\phi : E^{T'} \rightarrow E^N$ is a linear transformation that maps an T' -dimensional vector to a N -dimensional vector, summing the operating scales of x for each type of goods.

Since the positive elements of the technology matrix are standardized to 1, the sum of operating scales for each type of goods is equal to the output quantity of those goods. Therefore, we have:

$$\phi(x) = \mathbf{B}x \quad (5.2)$$

【定義 30Output Mapping: The mapping ϕ that translates the operating scale vector defined in (5.1) to the output quantity for each type of goods is referred to as the output mapping.

Given a vector $\phi(x) > 0$ denoted as x , we create the square matrix $\phi_x(\mathbf{A})$ by averaging the columns of \mathbf{A} by operating scale for each type of product. Define column $\phi_x(\mathbf{A})_{(n)}$ of matrix $\phi_x(\mathbf{A})$ as follows:

$$\phi_x(\mathbf{A})_{(n)} = \frac{\sum_{\gamma(\tau)=n} x_\tau a(\tau)}{\phi(x)_n}. \quad (5.3)$$

The same mapping ϕ_x is applied to \mathbf{B} and \mathbf{L} . ϕ_x is also a linear mapping.

【定義 31Technology-Goods Mapping: Given an operating scale vector x where the output quantity of all goods is positive, the matrix of technology \mathbf{B} , \mathbf{A} , and \mathbf{L} is aggregated for each goods by the mapping defined in (5.3), referred to as the technology-goods mapping.

Matrix $\phi_x(\mathbf{B})$ is equal to the identity matrix. This is because,

$$\phi_x(\mathbf{B})_{(n)} = \frac{\sum_{\gamma(\tau)=n} x_\tau b_{(\tau)}}{\phi(x)_n} = \frac{\sum_{\gamma(\tau)=n} x_\tau e_n}{\phi(x)_n} = \frac{\left(\sum_{\gamma(\tau)=n} x_\tau\right) e_n}{\phi(x)_n} = \frac{\phi(x)_n e_n}{\phi(x)_n} = e_n. \quad (5.4)$$

The second equality uses the fact that the n element of the country that produces goods n is equal to 1 and the other elements are 0.

From equation (5.4), it is evident that the $\phi_x(\mathbf{A})$ ($\phi_x(\mathbf{L})$) column of n represents the input quantities of each good (labor amount in each country) needed for the production of one unit of good n . Unlike \mathbf{L} , the column vectors of $\phi_x(\mathbf{L})$ can be positive in multiple countries, as they represent a weighted average of multiple technologies. From the meaning of Equation 637, Equation 638 holds. The left-hand side represents the input quantity of goods, while the right-hand side represents the input quantity necessary for each unit of production of goods 1, where Equation 613, and Equation 640 represent the output quantity. The same logic applies to Equations 134 and 135. In summary, we have the following:

$$\mathbf{A}x = \phi_x(\mathbf{A}) \phi(x), \quad (5.5)$$

$$\mathbf{B}x = \phi_x(\mathbf{B}) \phi(x), \quad (5.6)$$

$$\mathbf{L}x = \phi_x(\mathbf{L}) \phi(x). \quad (5.7)$$

【命題 20Existence of Inverse of Net Output Matrix by Commodity: The net output matrix by commodity as in Equation 643 has an inverse.

Proof: $\mathbf{I} - \phi_x(\mathbf{A})$ can be expanded into a product of two matrices.

$$\mathbf{I} - \phi_x(\mathbf{A}) = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ & \end{bmatrix}_{N \times 2N} \begin{bmatrix} \mathbf{I} \\ \phi_x(\mathbf{A}) \end{bmatrix}_{2N \times N}. \quad (5.8)$$

The first matrix on the right-hand side contains at least N linearly independent column vectors, thus its rank is at least N , but since it has N rows, the rank is N . The second matrix contains at least N linearly independent row vectors, so its rank is at least N , but with N columns, its rank is N . The product of these two matrices, denoted as $\mathbf{I} - \phi_x(\mathbf{A})$, has a rank of N . Therefore, $\mathbf{I} - \phi_x(\mathbf{A})$ is invertible. (End of proof)

Next, we seek the conditions under which $(\mathbf{I} - \phi_x(\mathbf{A}))^{-1}$ becomes a non-negative matrix. From (5.7), we have

$$(\phi_x(\mathbf{B}) - \phi_x(\mathbf{A})) \phi(x) = (\mathbf{B} - \mathbf{A}) x. \quad (5.9)$$

It should be noted that the square matrix $\phi_x(\mathbf{A})$ is a non-negative matrix. With this observation, if the pair (\mathbf{A}, \mathbf{B}) , $(x, (\mathbf{B} - \mathbf{A})x > 0)$ is $\phi_x(\mathbf{B}) - \phi_x(\mathbf{A}) = \mathbf{I} - \phi_x(\mathbf{A})$, then $\phi_x(\mathbf{B}) - \phi_x(\mathbf{A}) = \mathbf{I} - \phi_x(\mathbf{A})$ is non-negative invertible (Theorem II.1, p. 67, Fukakusa, 1961).

The main diagonal element of $(\mathbf{I} - \phi_x(\mathbf{A}))^{-1}$ is positive. Let's demonstrate this. Denoting the elements $(\mathbf{I} - \phi_x(\mathbf{A}))$ and $((\mathbf{I} - \phi_x(\mathbf{A})))^{-1}$ of (i, j) as c_{ij} , d_{ij} , it follows that if $k \neq n$, then $c_{nk} \leq 0$ holds.

$$1 = (\mathbf{I} - \phi_x(\mathbf{A})) (\mathbf{I} - \phi_x(\mathbf{A}))^{-1} \mathcal{O}(n, n) \text{element} \quad (5.10)$$

$$= \sum_{k=1}^N c_{nk} d_{kn} = \sum_{k \neq n} c_{nk} d_{kn} + c_{nn} d_{nn}. \quad (5.11)$$

If $k \neq n$ holds, then $c_{nk} \leq 0$ implies that the first term on the fourth side is non-positive. If d_{nn} equals 0, the second term on the fourth side becomes 0, and the entire fourth side is non-positive and not equal to 1. Therefore, $d_{nn} > 0$ holds. These points are summarized as propositions.

【定義 32 Scale of operation and net output matrix pair being productive, productive

activity scale: If $(\mathbf{B} - \mathbf{A})x > 0, x \geq 0$, holds, then the pair of operating scale x and net output matrix $\mathbf{B} - \mathbf{A}$ or simply referred to as pair $(x, \mathbf{B} - \mathbf{A})$ is productive. Here, x is called the productive activity scale.

【命題 21 Necessary and Sufficient Conditions for a Pair $(x, \mathbf{B} - \mathbf{A})$ to Be Productive and Properties of the Inverse Matrix:

1. The necessary and sufficient condition for a pair $(x, \mathbf{B} - \mathbf{A})$ to be productive is that the net output matrix per goods $\phi_x(\mathbf{B} - \mathbf{A})$ is non-negative reversibly invertible,
2. The main diagonal element of the inverse matrix $\phi_x(\mathbf{B} - \mathbf{A})^{-1}$ is positive.

Proof: We have already proved the necessity of 1 and 2. We now proceed to prove the sufficiency of 1. The goal is to show that there exists a solution $x \geq 0$ such that $(\mathbf{B} - \mathbf{A})x > 0$. For any x that results in $\phi(x) > 0$, we can define ϕ_x . If $\phi_x(\mathbf{B} - \mathbf{A})$ is non-negative reversibly invertible, it suffices to show that $(\mathbf{B} - \mathbf{A})x > 0$ holds for this x . Since $\phi_x(\mathbf{B} - \mathbf{A})$ is non-negative reversibly invertible, it implies that $\phi_x(\mathbf{B} - \mathbf{A})$ is regular. Therefore, the equation

$$\phi_x(\mathbf{B} - \mathbf{A}) \phi(x) = \mathbf{1}_N \quad (5.12)$$

has a solution. From Proposition , the **Price-Cost Equality Conditions for Each Goods** states that for a value vector $q' = (p' w')$,

$$p'(\mathbf{B} - \mathbf{A}) - w' \mathbf{L} = 0. \quad (5.13)$$

If the output quantity $\phi(x)$ is positive for any operating scale vector x , then for pair $(x, \mathbf{B} - \mathbf{A})$ being productive,

$$p' \phi_x(\mathbf{B} - \mathbf{A}) - w' \phi_x(\mathbf{L}) = 0. \quad (5.14)$$

Furthermore, if pair $(x, \mathbf{B} - \mathbf{A})$ is productive, we have

$$p' = -w' \phi_x(\mathbf{L}) \phi_x(\mathbf{B} - \mathbf{A})^{-1}. \quad (5.15)$$

Proof: Taking the weighted average across goods in column of (5.13) yields (5.14). The proposition in (5.15) states (see Proposition 21).

At a certain operating scale represented by x^* , even if pair $(x^*, \mathbf{B} - \mathbf{A})$ is productive, it does not guarantee that all combinations of inputs $x > 0$ lead to productivity. In other words, while a specific point on the technology face F may be productive, not all points on the open technology face $\text{int} F$ necessarily result in productivity. However, if a pair $(x^*, \mathbf{B} - \mathbf{A})$ is productive at a certain x^* , then for any set of inputs $c \geq 0$, there exists an operating scale $x \geq 0$ that allows for productivity. Let's demonstrate this.

To begin, let's establish that output quantity z can be achieved by defining the operating scale x for any given set of inputs z as follows:

$$X_\tau = X_\tau^* \frac{Z_{\gamma(\tau)}}{\phi(X_\tau^*)_{\gamma(\tau)}}. \quad (5.16)$$

The implication of this equation is to determine the level of operating scale for the technology producing goods n such that it maintains the ratio of operating scale between the production of goods n at operating scale x^* , while ensuring that the output quantity of goods n becomes z_n .

Let's actually calculate $\mathbf{B}x$.

$$\mathbf{B}x = \sum_{\tau} b_{(\tau)} x_\tau^* \frac{z_{\gamma(\tau)}}{\phi(x^*)_{\gamma(\tau)}} \quad (5.17)$$

$$= \sum_{\tau} e_{\gamma(\tau)} x_\tau^* \frac{z_{\gamma(\tau)}}{\phi(x^*)_{\gamma(\tau)}} \quad (\text{The output level of } b_{(\tau)} \text{ is normalized to 1}) \quad (5.18)$$

$$= \sum_{n=1}^N \sum_{\gamma(\tau)=n} e_{\gamma(\tau)} x_\tau^* \frac{z_{\gamma(\tau)}}{\phi(x^*)_{\gamma(\tau)}} \quad (\text{The sum is divided by goods}) \quad (5.19)$$

$$= \sum_{n=1}^N \sum_{\gamma(\tau)=n} e_n x_\tau^* \frac{z_n}{\phi(x^*)_n} \quad (\text{Substitute } \gamma(\tau) = n) \quad (5.20)$$

$$= \sum_{n=1}^N e_n \left(\sum_{\gamma(\tau)=n} x_\tau^* \right) \frac{z_n}{\phi(x^*)_n} \quad (\text{Move the part independent of } \tau \text{ outside}) \quad (5.21)$$

$$= \sum_{n=1}^N e_n \phi(x^*)_n \frac{z_n}{\phi(x^*)_n} \quad (\text{Using the definition of } \phi(x^*)) \quad (5.22)$$

$$= \sum_{n=1}^N e_n z_n = z. \quad (5.23)$$

$(\mathbf{B} - \mathbf{A})x = \phi_{x^*}(\mathbf{B} - \mathbf{A})z$ can also be proven. In the same x , it can also be demonstrated that the net output quantity becomes $\phi_{x^*}(\mathbf{B} - \mathbf{A})z$. Translating the Japanese economics paper written in LaTeX into English, we get:

$$\sum_{\tau=1}^{T'} (b_{i(\tau)} - a_{i(\tau)}) x_{\tau} = \sum_{\tau=1}^{T'} (b_{i(\tau)} - a_{i(\tau)}) \frac{x_{\tau}^*}{\phi(x^*)_{\gamma(\tau)}} z_{\gamma(\tau)} \quad (5.24)$$

$$= \sum_{\tau=1}^{T'} b_{i(\tau)} \frac{x_{\tau}^*}{\phi(x^*)_{\gamma(\tau)}} z_{\gamma(\tau)} - \sum_{\tau=1}^{T'} a_{i(\tau)} \frac{x_{\tau}^*}{\phi(x^*)_{\gamma(\tau)}} z_{\gamma(\tau)} \quad (5.25)$$

$$= \sum_{n=1}^N \sum_{\gamma(\tau)=n} b_{i(\tau)} \frac{x_{\tau}^*}{\phi(x^*)_{\gamma(\tau)}} z_{\gamma(\tau)} - \sum_{n=1}^N \sum_{\gamma(\tau)=n} a_{i(\tau)} \frac{x_{\tau}^*}{\phi(x^*)_{\gamma(\tau)}} z_{\gamma(\tau)} \quad (5.26)$$

$$= \sum_{n=1}^N \phi_{x^*}(\mathbf{B})_n z_n - \sum_{n=1}^N \phi_{x^*}(\mathbf{A})_n z_n \quad (5.27)$$

$$= \phi_{x^*}(\mathbf{B} - \mathbf{A}) z. \quad (5.28)$$

Setting the output quantity to $z = \phi_{x^*}(\mathbf{B} - \mathbf{A})^{-1}c$, the net output quantity c can be achieved. When the actual equation $z = \phi_{x^*}(\mathbf{B} - \mathbf{A})^{-1}c$ is substituted into (5.28), we get

$$\phi_{x^*}(\mathbf{B} - \mathbf{A}) z = \phi_{x^*}(\mathbf{B} - \mathbf{A}) \phi_{x^*}(\mathbf{B} - \mathbf{A})^{-1}c = c. \quad (5.29)$$

This proves that, if pair $(x^*, \mathbf{B} - \mathbf{A})$ is productive, there exists an operating scale that can achieve any non-negative net output $c \geq 0$. The productivity of pair $(x^*, \mathbf{B} - \mathbf{A})$ and the existence of an operating scale that achieves a positive net output quantity $c > 0$ are equivalent.

Conversely, if there exists an operating scale that can achieve any non-negative net output $c \geq 0$, it is trivial that the technology face is productive. Therefore,

【命題 22 Equivalent Conditions for Productive Technology Face: For any technology face F , the following conditions are equivalent.

1. There exists a point y such that $c > 0$ for some $y_G = c$ in technology face F .
2. For any given $c \geq 0$, the point where the technology face F holds is denoted by y .

【定義 33 Productive Technology Face: A technology face that satisfies the conditions of Proposition 22 is referred to as *productive* or *productive technology face*.

5.2 Meaning of Each Matrix

Explanation of the meaning of each matrix.

The N square matrix of size $\phi_x(\mathbf{A})$ has columns corresponding to the input quantities of each goods required to produce one unit of goods n .

The N square matrix of size $\phi_x(\mathbf{B} - \mathbf{A})^{-1}$ has columns corresponding to the quantities of each goods, both directly and indirectly, required to produce one unit of goods n . Let us denote such output quantities as vector $z_{(n)}$, which satisfies the following equation. The Japanese economic paper written in \LaTeX is translated into English as follows:

$$e_n = z_{(n)} - \phi_x(\mathbf{A}) z_{(n)}. \quad (5.30)$$

From Eq. (5.30),

$$e_n = \phi_x(\mathbf{B}) z_{(n)} - \phi_x(\mathbf{A}) z_{(n)} = \phi_x(\mathbf{B} - \mathbf{A}) z_{(n)}. \quad (5.31)$$

The first equation utilized Eq. (5.4) with $\mathbf{I} = \phi_x(\mathbf{B})$. Equation (5.31) shows that $z_{(n)}$ is the $\phi_x(\mathbf{B} - \mathbf{A})^{-1}$'s n column.

For a matrix of type M with \times rows and N columns, the n column of matrix $\phi_x(\mathbf{L})$ represents the labor input quantity of each country when producing one unit of goods n . Countries with positive elements in the n column of matrix $\phi_x(\mathbf{L})$ are producing goods n , and these countries are in a competitive state in the market for goods n . The elements of a matrix M with \times rows and N columns can be defined as follows:

$$\phi_x(\mathbf{L}) \text{'s } m \text{ rows} \times \phi_x(\mathbf{B} - \mathbf{A})^{-1} \text{'s } n \text{ columns} \quad (5.32)$$

$$= (\text{the labor input of each of the } m \text{ countries required to produce 1 unit of each goods}) \quad (5.33)$$

$$\times (\text{the output quantity of each goods required to produce only } n \text{ goods of 1 unit}) \quad (5.34)$$

Hence, it represents the labor input of m countries required to produce only goods n in one unit.

【定義 34 Embodied labor matrix in commodities and embodied labor vector : We refer to $\phi_x(\mathbf{L}) \phi_x(\mathbf{B} - \mathbf{A})^{-1}$ as the embodied labor matrix in commodities, and its n columns as the labor vector embodied in goods n (embodied labor).

When the labor vector embodied in goods n is positive, the wages of a country change, the prices of goods n also change. Even if the wages of countries with elements equal to zero change, the prices of goods n do not change.

It is also important to explain the meaning of face F towards technology $c_{(\tau)} = [g'_{(\tau)} \quad -l'_{(\tau)}]'$ that will be used later. This vector represents, for each country, the value obtained by deducting the labor directly inputted from the labor vectors embodied in commodities outputted by technology τ to technology $\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} g_{(\tau)} - l_{(\tau)}$. Expanding this vector yields

$$\text{The term } \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} g_{(\tau)} - l_{(\tau)} \quad (5.35)$$

$$= \sum_{n=1}^N \left(\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \right)_{(n)} g_{n(\tau)} - l_{(\tau)} \quad (5.36)$$

$$= \left(\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \right)_{(\gamma(\tau))} b_{\gamma(\tau)(\tau)} - \sum_{n=1}^N \left(\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \right)_{(n)} a_{n(\tau)} - l_{(\tau)} \quad (5.37)$$

.In the final equation, the first term represents the labor vector embodied in commodities produced by the technology, which is positive. The second term is a negative deduction item representing the intermediate labor vector embodied in commodities, and the third term also represents the directly inputted amount of labor, which is also negative. The labor vector embodied in technology can be interpreted as a "transformation function" that takes negative labor vector input to produce goods equivalent to a positive labor vector embodied. It can also be seen as representing the inverse transformation by multiplying it by -1. The product of this vector and the wage vector represents the profits of the technology.

【定義 35Technology Labor Matrix Embodied and Labor Vector Embodied: We call $\phi_x(\mathbf{L}_F)\phi_x(\mathbf{G}_F)^{-1}\mathbf{G}_F - \mathbf{L}_F$ the technology labor matrix embodied and each column $\phi_x(\mathbf{L}_F)\phi_x(\mathbf{G}_F)^{-1}g_{(\tau)} - l_{(\tau)}$ the labor vector embodied of technology (τ).

6 Logic of Value Determination

This paper assumes that in most cases, the technology face under consideration is productive, meaning that the international economy is in a state where all goods can be produced net of inputs. If the technology face is productive, then as per the proposition in Section 25 below in , the dimension of the technology face is at least N , while the dimension of the value face is at most M . In this paper, the model assumes that the price of goods is determined by costs, and among the costs, the cost of intermediate goods also depends on the price of goods. The only exogenous variable in costs is wages. If the dimension of the technology facet is N or higher, the dimension of the value facet is M or lower. It is possible to treat all or part of wages as an exogenous variable and the rest as endogenous variables to conform to the essence of the model.

Consider any r -dimensional technology facet with F (where $r \geq N$). Let the frame matrix of F be \mathbf{F} , the commodity component be $\mathbf{G}_F := \mathbf{B}_F - \mathbf{A}_F$, the labor component be \mathbf{L}_F , and the operating scale be x .

【命題 23】Equivalent Solution Sets: If the dimension $\phi_x(\mathbf{G}_F)$ of the r -dimensional technology facet F (where $r \geq N$) is regular, then the solution set of the following three simultaneous equations is equivalent. System of Equations 1:

$$P'\mathbf{G}_F - W'\mathbf{L}_F = 0. \quad (6.1)$$

System of Equations 2:

$$P'\phi_x(\mathbf{G}_F) - W'\phi_x(\mathbf{L}_F) = 0, \quad (6.2)$$

$$W' \left(\phi_x(\mathbf{L}_F)\phi_x(\mathbf{G}_F)^{-1}\mathbf{G}_F - \mathbf{L}_F \right) = 0. \quad (6.3)$$

System of Equations 3: Let \mathbf{F} denote the submatrix that represents the technology for producing each good, excluding one column for each good from the technology matrix, denoted by $\mathbf{F}_{r-N} = \begin{bmatrix} \mathbf{G}'_{r-N} & \mathbf{L}'_{r-N} \end{bmatrix}'$.

$$p' \phi_x(\mathbf{G}_F) - w' \phi_x(\mathbf{L}_F) = 0, \quad (6.4)$$

$$w' \left(\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{r-N} - \mathbf{L}_{r-N} \right) = 0. \quad (6.5)$$

Proof: We prove "If there exists a solution to the first set of equations, then there exists a solution to the second set of equations" by showing that applying technology-goods mapping f to (6.1) leads to the satisfaction of (6.2). By assumption, f is regular and has an invertible matrix such that

$$p' = w' \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \quad (6.6)$$

Substituting (6.6) into (6.1) yields (6.3) is satisfied.

We similarly demonstrate "If there exists a solution to the second set of equations, then there exists a solution to the first set of equations" by...Substituting any arbitrary p and w satisfying equation (6.2) into equation (6.3) yields:

$$\begin{aligned} 0 &= w' \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F - w' \mathbf{L}_F \\ &= p' \phi_x(\mathbf{G}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F - w' \mathbf{L}_F \\ &= p' \mathbf{G}_F - w' \mathbf{L}_F \end{aligned} \quad (6.7)$$

Hence, it is evident that equation (6.1) is satisfied.

The implication "Solution to system of equations 2 \Rightarrow Solution to system of equations 3" is straightforward.

To prove "Solution to system of equations 3 \Rightarrow Solution to system of equations 2": Since equation (6.2) is the same as equation (6.4), it suffices to show that there exists a w satisfying equation (6.5) that also satisfies equation (6.3). Without loss of generality, we may assume that the first N columns of the technology matrix \mathbf{F} sequentially produce goods1, goods2, ..., goods N (with the option to relabel technology numbers if necessary). According to the definition in the Production Mapping [Definition 30] ,

$$\phi_x(\mathbf{F}) = \mathbf{F}\mathbf{X} = \mathbf{F}_1\mathbf{X}_1 + \mathbf{F}_2\mathbf{X}_2 \quad (6.8)$$

this can be expressed. Here, the matrix \mathbf{X} is of type $T' \times N$, and its n column represents the operating scale weights for goods specified in equation (5.3). \mathbf{X}_1 and \mathbf{X}_2 are the first N rows and the remaining $T' - N$ rows of matrix \mathbf{X} , respectively. Since \mathbf{X}_1 is a diagonal matrix with non-zero elements, it is invertible. \mathbf{F}_1 and \mathbf{F}_2 are the first N columns and the remaining $T' - N$ columns of matrix \mathbf{F} , respectively. The rank of \mathbf{F}_1 is N . While \mathbf{F}_2 represents the third equation system, it is denoted with a subscript of 2 here to match the notation of \mathbf{X}_1 and \mathbf{X}_2 . Solving for \mathbf{F}_1 , we obtain

$$\mathbf{F}_1 = \phi_x(\mathbf{F}) \mathbf{X}_1^{-1} - \mathbf{F}_2 \mathbf{X}_2 \mathbf{X}_1^{-1}. \quad (6.9)$$

Expressing equation (6.9) in terms of commodity component and labor component, we have

$$\mathbf{G}_{F_1} = \phi_x(\mathbf{G}_F) \mathbf{X}_1^{-1} - \mathbf{G}_{F_2} \mathbf{X}_2 \mathbf{X}_1^{-1}, \quad (6.10)$$

$$\mathbf{L}_{F_1} = \phi_x(\mathbf{L}_F) \mathbf{X}_1^{-1} - \mathbf{L}_{F_2} \mathbf{X}_2 \mathbf{X}_1^{-1}. \quad (6.11)$$

Substituting equation (6.11) into the first term of labor matrix embodied, $\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F$, we have

$$\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F, \quad (6.12)$$

$$= \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} [\mathbf{G}_{F_1} \quad \mathbf{G}_{F_2}], \quad (6.13)$$

$$= \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} [\phi_x(\mathbf{G}_F) \mathbf{X}_1^{-1} - \mathbf{G}_{F_2} \mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{G}_{F_2}], \quad (6.14)$$

$$= \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} ([\phi_x(\mathbf{G}_F) \mathbf{X}_1^{-1} \quad \mathbf{0}] + [-\mathbf{G}_{F_2} \mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{G}_{F_2}]), \quad (6.15)$$

$$= [\phi_x(\mathbf{L}_F) \mathbf{X}_1^{-1} \quad \mathbf{0}] + \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{F_2} [-\mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{I}_{T'-N}]. \quad (6.16)$$

Substituting the second term \mathbf{L}_F of labor matrix embodied into equation (6.19), we get

$$\mathbf{L}_F = [\mathbf{L}_{F_2} \quad \mathbf{L}_{F_2}] \quad (6.17)$$

$$= [\phi_x(\mathbf{L}_F) \mathbf{X}_1^{-1} - \mathbf{L}_{F_2} \mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{L}_{F_2}] \quad (6.18)$$

$$= [\phi_x(\mathbf{L}_F) \mathbf{X}_1^{-1} \quad \mathbf{0}] + \mathbf{L}_{F_2} [-\mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{I}_{T'-N}]. \quad (6.19)$$

Substituting equations (6.16) and (6.19) into labor matrix embodied, we have

$$\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F - \mathbf{L}_F \quad (6.20)$$

$$= \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{F_2} [-\mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{I}_{T'-N}] - \mathbf{L}_{F_2} [-\mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{I}_{T'-N}] \quad (6.21)$$

$$= (\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{F_2} - \mathbf{L}_{F_2}) [-\mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{I}_{T'-N}]. \quad (6.22)$$

By multiplying the wage w that satisfies equation (6.5) from the left,

$$w' (\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F - \mathbf{L}_F) \quad (6.23)$$

$$= w' (\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{F_2} - \mathbf{L}_{F_2}) [-\mathbf{X}_2 \mathbf{X}_1^{-1} \quad \mathbf{I}_{T'-N}] \quad (6.24)$$

$$= 0 \quad (6.25)$$

Thus, it has been demonstrated that "the solution of the simultaneous equations set 3 implies the solution of the simultaneous equations set 2." (End of proof)

The following proposition is an extraction of (6.6) from the proof of Proposition 23:

【命題 24Explicit Expression of Prices: If $\phi_x(\mathbf{G}_F)$ is regular, then for the solution set of three simultaneous equations,

$$p = w' \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \quad (6.26)$$

holds.

Proposition 24 implies that the goods price is proportional to the embodied labor costs of goods.

The condition for technology to be productive is that $\phi_x(\mathbf{G}_F)$ is regular.

【命題 25Properties of Productive Technology Face: If a r -dimensional technology face, where F (with $r \geq N$), is productive, then:

1. In the solution set of the three simultaneous equations (Equations (6.3), (6.4), and (6.5)), Equation (6.26) holds.
2. The rank of \mathbf{G}_F is N .
3. The rank of \mathbf{F} is at least N .
4. All rows of \mathbf{G}_F and some rows of \mathbf{L}_F can form pairs of r linearly independent vectors.

Proof: From Proposition 21, $\phi_x(\mathbf{G}_F)$ is non-negative invertible, hence regular. Therefore, fulfilling the conditions of Proposition 24, the assertion follows from Proposition 24.

To prove 2 by contradiction: Translated text:

Since technology face F is productive, from Proposition 21, $\phi_x(\mathbf{G}_F)$ is regular. That is, $\text{rank}\{\phi_x(\mathbf{G}_F)\} = N$. Assuming that the row vectors of \mathbf{G}_F are not linearly independent ($\text{rank}\{\mathbf{G}_F\} < N$), there exists a nonzero vector q such that $q'\mathbf{G}_F = 0$. This leads to a contradiction with $\text{rank}\{\phi_x(\mathbf{G}_F)\} = N$ when both sides are mapped by mapping ϕ_x to the same vector q , resulting in $q'\phi_x(\mathbf{G}_F) = 0$.

3 is trivial.

Proof of 4: Since the rank of \mathbf{F} is r , the r row vectors of \mathbf{G}_F (these are linearly independent by 2) can be supplemented by adding a suitable number of row vectors from \mathbf{L}_F to make the total r row vectors linearly independent. (End of proof)

【命題 26Rank of technology labor matrix embodied : Technology face F is of dimension r (where $r \geq N$), and $\phi_x(\mathbf{G}_F)$ is assumed to be regular. The following holds:

1. The rank of the technology labor matrix embodied in Equation 782 is $r - N$.
2. Let the sub-technology matrix representing the remaining technologies after excluding one production technique (column vector) for each good from the technology matrix in Equation 247 be denoted as \mathbf{F}_{r-N} . The rank of the labor matrix embodied in $\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{r-N} - \mathbf{L}_{r-N}$ of \mathbf{F}_{r-N} is also $r - N$.

Proof: Let the set of solutions to the system of equations 1 be denoted as Q , and the set

of solutions to equation 3 in (6.3) be denoted as W . Q is a set of solutions representing a $(M + N) \times M$ -type matrix

$$\mathbf{X} := \begin{bmatrix} \phi_x(\mathbf{G}_F)^{\prime -1} \phi_x(\mathbf{L}_F)' \\ \mathbf{I}_M \end{bmatrix} \quad (6.27)$$

representing the image of W . That is, $Q = \mathbf{X}W$. From the form of \mathbf{X} , the kernel of \mathbf{X} is only 0 (since the x that satisfies $0 = \mathbf{X}x$ is only $x = 0$ by focusing on the \mathbf{I}_M part of \mathbf{X}). By applying Satake (1974) Lemma III.2 (page 104) gives:

$$d\{\mathbf{X}W\} + d\{\mathbf{X}^{-1}(0) \cap W\} = d\{W\} \Rightarrow \quad (6.28)$$

$$d\{Q\} = d\{W\} - d\{\mathbf{X}^{-1}(0) \cap W\} \quad (6.29)$$

$$= d\{W\} - d\{\{0\} \cap W\} \quad (6.30)$$

$$= d\{W\} - d\{\{0\}\} \quad (6.31)$$

$$= d\{W\}. \quad (6.32)$$

Therefore, the dimensions of the two subspaces in Equation 85 and Equation 890 are equal.

On the other hand, System of Equations 1 (Equation (6.1)) is equivalent to Equation 904, hence the solution set in Equation 85 is the kernel of matrix Equation 906. Applying Theorem III.7 (p. 104) by Sawu (1974), we have

$$d\{\mathbf{F}E^{M+N}\} + d\{\ker \mathbf{F}'\} = M + N \Rightarrow \quad (6.33)$$

$$\text{rank}\{\mathbf{F}\} + d\{Q\} = M + N. \quad (6.34)$$

Since $\text{rank}\{\mathbf{F}\} = r$,

$$d\{Q\} = M + N - r. \quad (6.35)$$

From equations (6.32) and (6.35), it follows that $d\{W\} = M + N - r$. Substituting $d\{W\} = M + N - r$ into the modified equations of (6.34) regarding \mathbf{F} and E^{M+N} as labor matrix embodied and E^M respectively,

$$\text{rank}\left\{\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F - \mathbf{L}_F\right\} = M - d\{W\} = M - (M + N - r) = r - N. \quad (6.36)$$

Thus, item has been proven. The application of (6.36) to the labor matrix embodied in \mathbf{F}_{r-N} in , along with the equivalence of the solution sets in (6.3) and (6.5 (Proposition 23), yields:

$$\begin{aligned} & \text{Rank}\left\{\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_{r-N})^{-1} \mathbf{G}_{r-N} - \mathbf{L}_F\right\} \\ &= M - d\{\text{Solution set of 6.1.5}\} \\ &= M - d\{W\} \\ &= M - (M + N - r) \\ &= r - N \end{aligned} \quad (6.37)$$

which concludes the proof.

【定義 36Price Associated with Wage: The price determined by (6.6) is referred to as the price accompanying wages.

The following proposition holds for the associated price.

【命題 27Determinants of the Associated Price Ratio Between Goods and Goods Price-Wage Ratio:

1. Different associated price ratios between goods and the ratio of associated prices to wages are determined by inter-country wage ratios.
2. In a closed economy of a single country where wages are chosen as the unit of value, prices are determined solely by technical conditions.

Proof: 1 follows from (6.6). Applying 1 to a closed economy of a single country leads to the truth of 2. (End of Proof)

Equation (6.6) merely determines the subspace, so if q is a solution, then $-q$ is also a solution. The condition that determines the sign is that q is a point of C^p . The wage face satisfying the inequality

$$w' \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G} - w' \mathbf{L} \leq 0 \quad (6.38)$$

is expressed as follows:

【命題 28Wage Face's Explicit Expression 1: The wage face $P_W(F)$ satisfies the following equations:

$$\begin{aligned} w' \left(\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_F - \mathbf{L}_F \right) &= 0, \\ w' \left(\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G} - \mathbf{L} \right) &\leq 0. \end{aligned} \quad (6.39)$$

Equation (6.39) serves not only as the determination formula for the wage face $\mathcal{P}_W(F)$ associated with the technology face F , but also illustrates that $\mathcal{P}_W(F)$ is a convex polyhedral cone. Generally, the eligibility of technology face F does not guarantee the presence of positive elements in wage face $\mathcal{P}_W(F)$.

The location of point $\mathcal{P}(F)$ on the face of $q' = (p', w')$ does not depend on which point of F is. From this, it is understood that in (6.6), both $\phi_x(\mathbf{G}_F)$ and $\phi_x(\mathbf{L}_F)$ depend on the operating scale x , but the product $\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1}$ does not depend on x due to the offsetting effects on the operating scale x .

【命題 29Uniqueness of the labor vector embodied in commodities: The labor vector embodied in commodities $\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1}$ does not depend on which point of the technology

face F it takes.

From (6.6) and Proposition 29,

【命题 30 Weak uniqueness of associated prices: Within the value face, the same price holds for the same wage.

For the case of $r < M + N - 1$, the dimension of the wage face is greater than 1. Let us delve into a more detailed analysis of the structure of the wage face. Revisiting (6.5):

$$w' \left(\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{r-N} - \mathbf{L}_{r-N} \right) = 0. \quad (6.40)$$

As elucidated in the definitions from Proposition 31 and Proposition 30, the matrix on the left-hand side of (6.40) corresponds to the sub-technology matrix $\phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1} \mathbf{G}_{r-N} - \mathbf{L}_{r-N}$ where, as discussed in the context of sub-technology matrices in the proof of Proposition 30 in Section \mathbf{F}_{r-N} , the rank of the labor matrix embodied in \mathbf{F}_{r-N} is $r - N$. The equation (6.40) indicates that the associated wages stipulated by the technology face F ensure that the product of the labor matrix embodied in \mathbf{F}_{r-N} and wages equals zero. The equation (6.40) defines a linear subspace of dimension $M + N - r$ within E^M . It is indicating that the solution lies in the null space of a coefficient matrix of rank $r - N$. The dimension of this null space is $M - (r - N) = M + N - r$.

【命题 31 Determination of Wages: Wages associated with the technology face in a r -dimensional labor matrix embodied such that the product with the technology face in F results in zero are determined. Wages have $M + N - r$ degrees of freedom.

Within equation (6.40), the rank of the technology labor matrix embodied is $r - N$, thus it is possible to extract pairs of $r - N$ linearly independent column vectors. Let us denote this $r - N$ -type matrix as \mathbf{E} (embodied labor). The rank of \mathbf{E} is $r - N$, indicating the existence of $r - N$ linearly independent row vectors. Without loss of generality, we can consider the first $r - N$ row vectors to be linearly independent (reordering if necessary). Let \mathbf{E}_I represent these first linearly independent $r - N$ rows, and \mathbf{E}_{II} represent the remaining $M + N - r$ rows. \mathbf{E}_I is a regular square matrix of order N . The corresponding wage vector is also denoted as w_I and w_{II} . Rewriting (6.40) in a partitioned matrix form, we have

$$w'_I \mathbf{E}_I + w'_{II} \mathbf{E}_{II} = 0. \quad (6.41)$$

To solve this equation.

****Proposition**** ****Explicit Expression of Wage Surface 2:**** The r -dimensional technology face with labor matrix embodied in the F -dimensional technology space (defined in Definition 35) can be represented by a submatrix of rank $r - N$ and with $(r - N)$ columns denoted by \mathbf{E} . This submatrix \mathbf{E} consists of the first $(r - N)$ linearly independent row vectors, denoted as \mathbf{E}_I ,

and the remaining $(M + N - r)$ row vectors denoted as \mathbf{E}_{II} . Correspondingly, the wage vectors are also represented as w_I and w_{II} . At points on the wage face of the technology face F , the following holds:

$$w'_I = -w'_{II} \mathbf{E}_{II} \mathbf{E}_I^{-1} \quad (6.42)$$

【命題 32】 Although we have explained the process of price determination and wage determination in two stages so far, it is also possible to explain value determination in a single stage. In that case, it suffices to solve the system of equations (6.1).

Assume that the r dimensional technology face is productive. Referring to Proposition 25, especially, it can be guaranteed that we can construct r pairs of linearly independent vectors from all rows of \mathbf{G}_F and some rows of \mathbf{L}_F . Without loss of generality, let the first $(r - N)$ rows of \mathbf{L}_F be such linearly independent row vectors, denoted as \mathbf{L}_{F_I} . Consider the matrix

$$\begin{bmatrix} \mathbf{G}_F \\ -\mathbf{L}_{F_I} \end{bmatrix} \quad (6.43)$$

whose rank is equal to r . Therefore, since there are r linearly independent columns, we can construct a r square matrix from them as follows:

$$\begin{bmatrix} \tilde{\mathbf{G}}_F \\ -\tilde{\mathbf{L}}_{F_I} \end{bmatrix} \quad (6.44)$$

This matrix is nonsingular, hence it possesses an inverse^{*8}.

The first $r - N$ rows of matrix \mathbf{L}_F are represented as \mathbf{L}_{F_I} , and the remaining $(M + N - r)$ rows are denoted as $\mathbf{L}_{F_{II}}$, with the same columns as in (6.44) identified as \mathbf{L}_{F_I} . The extracted matrix from $\mathbf{L}_{F_{II}}$ and $\tilde{\mathbf{L}}_{F_I}$ is denoted by $\tilde{\mathbf{L}}_{F_I}$ and $\tilde{\mathbf{L}}_{F_{II}}$. Corresponding to the rows of \mathbf{L}_{F_I} and $\mathbf{L}_{F_{II}}$, the wage vector w is partitioned into w_I, w_{II} . The price-cost equality equation can be expressed as:

$$(p' \ w'_I) \begin{bmatrix} \tilde{\mathbf{G}}_F \\ -\tilde{\mathbf{L}}_{F_I} \end{bmatrix} = w'_{II} \tilde{\mathbf{L}}_{F_{II}}. \quad (6.45)$$

Multiplying by the inverse matrix yields:

$$(p' \ w'_I) = w'_{II} \tilde{\mathbf{L}}_{F_I} \begin{bmatrix} \tilde{\mathbf{G}}_F \\ -\tilde{\mathbf{L}}_{F_I} \end{bmatrix}^{-1} \quad (6.46)$$

【命題 33 Explicit Expression of Values: The value of a productive r -dimensional technology facing uncertainty is given by the following equation with w_{II} as the exogenous variable.

^{*8} However, since non-diagonal elements may not be nonnegative, the inverse matrix may not necessarily be non-negative.

$$(P' \ W'_I) = -W'_{II} \tilde{\mathbf{L}}_{F_{II}} \begin{bmatrix} \tilde{\mathbf{G}}_F \\ -\tilde{\mathbf{L}}_{F_I} \end{bmatrix}^{-1} \quad (6.47)$$

Here, $\tilde{\mathbf{G}}_F, \tilde{\mathbf{L}}_{F_I}, \tilde{\mathbf{L}}_{F_{II}}, w_I, w_{II}$ represents the variables defined in (6.44) to (6.45).

7 Analysis of the Eligible Technology Face

7.1 Positive Wage Range

【定義 37 Positive Wage Range: The union set of wage faces in the positive domain associated with an eligible technology face of N or more dimensions.

$$\hat{W} := \bigcup_{F \text{ is an eligible technology face with } N \text{ or more dimensions}} \mathcal{V}_W(F) \quad (7.1)$$

is referred to as the positive wage range.

When considering technology faces of N or more dimensions, positive value does not exist unless the wage is at point \hat{W} .

It can be shown that eligible technology faces of N or more dimensions are productive (Definition 33). Select an eligible technology face of N or more dimensions, say F . There exists a positive price p such that it satisfies (6.1) for $w \in \mathcal{V}_W(F)$. According to the Theorem II.1 in Nikaido (1961, p. 67), $\mathbf{I} - \phi_x(\mathbf{A}_F)$ is nonnegative and reversible. Therefore, equation $(\mathbf{I} - \phi_x(\mathbf{A}_F))z = c$ has a non-negative solution z for any final demand satisfying c , which fulfills condition 1 within Proposition 22. According to Definition 33, the technology face F is productive. Hence,

【命題 34 Necessary Conditions and Properties of Eligible Technology Face: An eligible technology face in dimensions of N or higher is productive and possesses an operating scale to pure production for any non-negative final demand.

The converse of the first part of Proposition 34 is not necessarily true. We provide numerical examples of technology faces that are productive but not eligible. Table 1 presents a technology matrix with four technologies in the case of $M = 1, N = 2$.

表1 Numerical example of technology faces that are productive but not eligible

Technology	$c_{(1)}$	$c_{(2)}$	$c_{(3)}$	$c_{(4)}$
Goods 1	-0.5	1	2	-0.75
Goods 2	1	-0.5	-0.75	2
Labor	-1	-1	-1	-1

The technologically feasible set defined by $(c_{(1)} \ c_{(2)} \ c_{(3)} \ c_{(4)})$ consists of four faces:

$$(c_{(1)} \ c_{(2)}), (c_{(2)} \ c_{(3)}), (c_{(3)} \ c_{(4)}), (c_{(4)} \ c_{(1)}) \quad (7.2)$$

Out of these four technology faces, $(c_{(1)} \ c_{(2)})$ is productive. In fact,

$$\begin{aligned} y_G &:= 0.5c_{(1)} + 0.5c_{(2)} \\ &= 0.5(-0.5, 1)' + 0.5(1, -0.5)' \\ &= (0.25, 0.25)' > 0 \end{aligned} \quad (7.3)$$

which implies that $(c_{(1)} \ c_{(2)})$ can produce two goods. However, the value associated with $(c_{(1)} \ c_{(2)})$ is shown to be negative as indicated in Table 2, not positive. Assessing the benefits of technology $c_{(2)}$, $c_{(3)}$, $c_{(4)}$ evaluated at this value yields a negative outcome, confirming that this value represents the outward normal of the technology face $(c_{(1)} \ c_{(2)})$. Hence, while the technology face $(c_{(1)} \ c_{(2)})$ is productive, it is not eligible.

Summarizing the above as a proposition:

【命題 35A productive technology face is not necessarily an eligible technology face:

There exist numerical examples where a technology face is productive but not eligible. A productive technology face is not necessarily an eligible technology face.

表2 The value of technology face $(c_{(1)} \ c_{(2)})$ and profits of each technology

The value of technology face $(c_{(1)} \ c_{(2)})q$	p_1	p_2	w_1	
	-2	-2	-1	
Technology	$c_{(1)}$	$c_{(2)}$	$c_{(3)}$	$c_{(4)}$
Profits of each technology evaluated at q	0	0	-1.5	-1.5

The value associated with another productive technology face $(c_{(3)} \ c_{(4)})$, is positive, as shown in Table 3. Thus, technology face $(c_{(3)} \ c_{(4)})$, is an eligible technology face.

表3 Value of Technology Face $(c_{(3)} \ c_{(4)})$ and Profits of Each Technology

Value of technology face $(c_{(3)} \ c_{(4)})$	Technology p_1	Technology p_2	Technology w_1	
	0.8	0.8	1	
Technology	Technology $c_{(1)}$	Technology $c_{(2)}$	Technology $c_{(3)}$	Technology $c_{(4)}$
Profits of each technology evaluated at q	-0.6	-0.6	0	0

Figure 1 depicts a cross-section of this technologically feasible set truncated at $y_3 = -1$. From this figure, we can observe the following.

1. The technology face $(c_{(1)} \ c_{(2)})$ is productive; however, since it is possible to move north-east from any point on $(c_{(1)} \ c_{(2)})$, the points on this technology face are not maximal technologically feasible points (defined in Definition 29). Furthermore, the elements of the outward normal of this technology face are negative.
2. Another productive technology face, $(c_{(3)} \ c_{(4)})$, has the property that all points on it are maximal technologically feasible points. Additionally, the elements of the outward normal of this technology face are positive.
3. It is not necessarily the case that the net production of commodity components at all points on the technology face $(c_{(3)} \ c_{(4)})$ is positive.

Items 1 and 2 correspond to Proposition 17. Item 3 corresponds to the content of the paragraph just before Equation (5.16), where the analysis of equivalent conditions for productive technology faces begins. The uniqueness of associated prices with given wages

Considering that different eligible technology faces entail distinct labor and intermediate goods input conditions, it might seem that prices determined by Equation (6.1.6) would vary across technology faces for the same wage. However, it can be shown that when a wage lies in the intersection of two wage face positive regions, prices remain equal across different technology faces.

Let F_I and F_{II} represent two distinct eligible technology faces. Let $\mathcal{V}_W(F_I) \cap \mathcal{V}_W(F_{II}) \neq \emptyset$ denote a common wage. Take any $w \in \mathcal{V}_W(F_I) \cap \mathcal{V}_W(F_{II})$. Denote the prices determined by Equation (6.1.6) for the common w associated with technology faces F_I and F_{II} as p_I, p_{II} . Let $\mathbf{B}_i, \mathbf{A}_i, \mathbf{L}_i$, $i = I, II$, represent the output, input, and labor components of F_i . The following equation holds: The first equation represents the price-cost equalization condition, while the second inequality states that the profits of all technologies evaluated at the value surface region identified in p_{II} are non-positive, as indicated in Proposition 19, item 1. From these equations, multiplying both sides by a non-negative matrix $\phi_I(\mathbf{I} - \mathbf{A}_I)^{-1}$ yields

$$p_I \geq p_{II}. \quad (7.4)$$

By exchanging the roles of F_I and F_{II} , the same argument can be made, leading to

$$p_{II} \geq p_I. \quad (7.5)$$

Therefore,

$$p_I = p_{II}. \quad (7.6)$$

【命題 36 Strong Uniqueness of Associated Prices: When wages exist in the positive domain of the intersection of two eligible technology faces, prices satisfying the price-cost equality equations of the two technology faces are equal.

Proposition 36 asserts that even if the value faces are different, the same price holds for the same wage, indicating a stronger property than the statement in Proposition 30 that "within each value face, the same price holds for the same wage."

It is possible to determine the conditions under which two wage faces do not intersect with respect to a technology face. Using the same notation as above, let the set of common technology numbers belonging to both F_I and F_{II} be denoted as $T_{I \cap II}$, the set of technology numbers belonging only to F_I but not to F_{II} be denoted as $T_{I \setminus II}$. The following equation holds:

$$T_{I \cap II} = F_I \cap F_{II} \setminus T_{I \setminus II}$$

The translation of the provided text is as follows:

$$p'_I (\mathbf{B}_I - \mathbf{A}_I) = w' \mathbf{L}_I, \quad (7.7)$$

$$p'_{II} (\mathbf{B}_{I \cap II} - \mathbf{A}_{I \cap II}) = w' \mathbf{L}_{I \cap II}, \quad (7.8)$$

$$p'_{II} (\mathbf{B}_{I \setminus II} - \mathbf{A}_{I \setminus II}) < w' \mathbf{L}_{I \setminus II}. \quad (7.9)$$

Evaluate (7.8) at p_{II} and note that the technical profit contained in F_{II} is zero, and evaluate (7.9) at p_{II} to observe that the profit from technologies not included in F_{II} is negative (Proposition 19).

The image of (7.7) under ϕ_I is as follows. The set of goods produced by technology number $T_{I \setminus II}$ is denoted as $N_{I \setminus II}$, and the set of goods produced by technology number $T_{I \cap II}$ is denoted as $N_{I \cap II}$. There may be some goods in the product of technology $T_{I \cap II}$ that overlap with the product of technology $T_{I \setminus II}$, so the intersection of $N_{I \setminus II}$ and $N_{I \cap II}$ may not be empty (see Figure).

Taking the image of equations (7.8) and (7.9) under ϕ_I . Translated version:

If $N_{I \setminus II}$'s goods are produced by technology from $T_{I \cap II}$, the profit evaluated at p_{II} is zero, and the negative profit evaluated at $T_{I \setminus II}$'s technology is weighted averaged. Therefore, the profit of $N_{I \setminus II}$'s goods evaluated at p_{II} becomes negative. When n differ in price, as shown in Equations (), (), and (), the following relationships hold:

For n in $(p'_{II} - p'_I)_{(n)}$, we have:

$$\begin{aligned} (p'_{II} - p'_I)_n &= \left((p'_{II} - p'_I) (I - \phi_I (\mathbf{A}_I)) (I - \phi_I (\mathbf{A}_I))^{-1} \right)_n \\ &= (p'_{II} - p'_I) (I - \phi_I (\mathbf{A}_I)) (I - \phi_I (\mathbf{A}_I))_{(n)}^{-1} \\ &= \sum_{i \in N_{I \setminus II}} ((p'_{II} - p'_I) (I - \phi_I (\mathbf{A}_I)))_i (I - \phi_I (\mathbf{A}_I))_{i(n)}^{-1} \\ &\quad + \sum_{i \in N_{I \cap II} \setminus N_{I \setminus II}} ((p'_{II} - p'_I) (I - \phi_I (\mathbf{A}_I)))_i (I - \phi_I (\mathbf{A}_I))_{i(n)}^{-1} \\ &= \sum_{i \in N_{I \setminus II}} ((p'_{II} - p'_I) (I - \phi_I (\mathbf{A}_I)))_i (I - \phi_I (\mathbf{A}_I))_{i(n)}^{-1} \end{aligned} \quad (7.10)$$

If goods belonging solely to technology F_I are represented by goods from technology $n \in N_{I \setminus II}$, then we have:

$$p_{II \ n} < p_{I \ n}. \quad (7.11)$$

Similarly, if goods belonging solely to technology F_{II} are represented by goods from technology $n' \in N_{II \setminus I}$, then:

$$p_{I \ n'} < p_{II \ n'}. \quad (7.12)$$

This leads to a contradiction. If there exist goods that are the same across the products of technologies belonging exclusively to F_I and F_{II} , then the wage face positive regions of the two technology faces do not intersect. Taking the contrapositive, if the wage face positive regions intersect, all products of technologies exclusive to F_I and F_{II} must be different. Moreover, if not all goods are produced by technologies belonging to both F_I and F_{II} , then non-overlapping technologies will produce the remaining common goods. This implies that the wage face positive regions of the two technology faces do not intersect.

【命題 37 Intersection of Two Wage Faces: Represent two eligible technology faces as F_I and F_{II} .

1. If there are identical goods produced by technologies exclusive to F_I and F_{II} , the positive regions of the two technology faces' wage faces do not intersect.
2. If the positive regions of the two wage faces intersect, then products exclusive to F_I and F_{II} are all distinct.
3. If overlapping technologies of F_I and F_{II} do not produce all goods ($N_{I \cap II} \subsetneq N$), then the positive regions of the two technology faces' wage faces do not intersect.

7.2 Minimal Wage Face and Maximal Technology Face

The intersection $W_1 \cap W_2$ of two wage faces W_1 and W_2 is also a wage face. Let us prove this. Denote the original value faces of W_1 and W_2 as V_1 and V_2 . $V_1 \cap V_2$ is a value face (Proposition 1). In any $q = (p' w')' \in V_1 \cap V_2$ of a Japanese economics paper written in LaTeX, the wage component w belongs to both W_1 and W_2 . Thus, the wage component of $V_1 \cap V_2$ is included in $W_1 \cap W_2$. Conversely, for any $w \in W_1 \cap W_2$, there exist p_1 and p_2 such that $(p'_1 w')' \in V_1$ and $(p'_2 w')' \in V_2$ hold. However, by Proposition 36 (the strong uniqueness of associated prices), $p_1 = p_2 =: p$, and therefore $(p' w')' \in V_1 \cap V_2$. This implies that $W_1 \cap W_2$ is included in the wage component of $V_1 \cap V_2$. Thus, it has been proven that the wage component of the value face $V_1 \cap V_2$ is equal to the intersection of two wage faces, $W_1 \cap W_2$. Consequently, the definition of the minimal wage face containing $w \in \hat{W}$ can be established.

【定義 38 Minimal wage face: The minimal wage face containing wage $w \in \hat{W}$ with wage w is

referred to as the minimal wage face and denoted by $W(w)$.

$$W(w) := \bigcap_{w \in P_W(F)} P_W(F) \quad (7.13)$$

The mapping of the wage component equal to $W(w)$ in value face V is a technology face, as shown in the anti-isomorphism in Proposition 5 item (ii). Therefore, the following definition holds valid.

【定義 39Maximal Technology Face: The image of the value face V with $V_W = W(w)$ is denoted as the maximal technology face of wage w .

Furthermore,

【命題 38Properties of Maximal Technology Face: The maximal technology face is the largest among the technology faces where the wage face equals the minimal wage face $W(w)$.

Proof: For any $V_W \supset W(w)$,

$$C(w) \supset \mathcal{P}(V). \quad (7.14)$$

(End of proof)

Since the maximal technology face is defined with respect to $w \in \hat{W}$,

【命題 39Eligibility of the maximal technology face: The maximal technology face $C(w)$ is eligible as a technology face.

7.3 Convergence of iterative price adjustments for given wages^{*9}

We examine the outcomes of decentralized pricing actions of firms in a market economy. The movement of quantity faces parallel to price adjustments is not explicitly addressed here. If price adjustments and quantity face movements are independent, the conclusions here are compatible with any quantitative movements.

Let w be any wage within the positive wage range \hat{W} that is given. Each firm repeatedly adjusts prices at discrete time intervals. Both prices and wages are denominated in a common currency. The adjustment process proceeds as follows: Given its own wage and the previous goods prices set by all firms, each firm calculates its production cost for its own goods. Although production costs differ by technology even for the same goods, the lowest production cost serves as the current price for this round. When considering the original technology matrix, the price is set by adding a given positive markup rate. Firms without the least-cost technology either exit the market or accept a lower markup rate, which does not affect the subsequent formulation.

^{*9} The convergence to the associated price through iterative price adjustments for given wages was first proved in Shiozawa, Y Chapter 2 Th.4.4.10 (p. 82) in Shiozawa, Y., Morioka, M., & Taniguchi, K. (2019). This section is indebted to that description. The book does not mention the convergence to the maximal wage face.

Formulating the process, it is as follows, starting from the given $q(0)' = \begin{bmatrix} p'(0) & w' \end{bmatrix}$. The revised price at the t th iteration ($t \geq 1$) is determined by the following equation:

$$p(t)_n = \min_{\gamma(\tau)=n} (p(t-1)'a_{(\tau)} + w'l_{(\tau)}) \quad (7.15)$$

With infinitely repeated price adjustments, prices converge to a certain level, and the collective choice of technologies by firms essentially aligns with a frame matrix of an eligible technology face. Let us now demonstrate this.

The given wage $w \in \hat{W}$ is denoted by F , representing the maximal technology face $C(w)$. Equation 247 is considered an eligible technology face (see Proposition 39), thus it is productive (see Proposition 34). The technology matrix of Equation 247 is represented by Equations 1196, 1197, and 854. Equation 1196 is assumed to be normalized by output quantities. The associated price of Equation 159 is denoted by Equation 1201.

$$p'_w \mathbf{B}_F = p'_w \mathbf{A}_F + w' \mathbf{L}_F. \quad (7.16)$$

The iterative price adjustment formula in Equation (7.4.1) also holds for the technology face of Equation 247. In matrix form, the economic model can be expressed as:

$$p'(t) \mathbf{B}_F \leq p'(t-1) \mathbf{A}_F + w' \mathbf{L}_F. \quad (7.17)$$

The image under mapping ϕ_x of equations (7.16) and (7.17) yields:

$$p'_w = p'_w \phi_x(\mathbf{A}_F) + w' \phi_x(\mathbf{L}_F), \quad (7.18)$$

$$p'(t) \leq p'(t-1) \phi_x(\mathbf{A}_F) + w' \phi_x(\mathbf{L}_F). \quad (7.19)$$

Subtracting equation (7.18) from equation (7.19), we obtain:

$$\begin{aligned} p'(t) - p'_w &\leq (p'(t-1) - p'_w) \phi_x(\mathbf{A}_F), \\ p'(t-1) - p'_w &\leq (p'(t-2) - p'_w) \phi_x(\mathbf{A}_F). \end{aligned} \quad (7.20)$$

Multiplying the non-negative square matrix $\phi_x(\mathbf{A}_F)$ to both sides of the second equation does not change the direction of the inequality. By substituting the result into the first equation, we obtain

$$P'(t) - P'_w \leq (P'(t-2) - P'_w) \Phi_x(\mathbf{A}_F)^2. \quad (7.21)$$

Iterating this process leads to the following equation:

$$P'(t) - P'_w \leq (P'(0) - P'_w) \Phi_x(\mathbf{A}_F)^t. \quad (7.22)$$

Since technology factor F is productive, $\mathbf{I} - \phi_x(\mathbf{A}_F)$ is non-negative revocable. This implies $\lim_{t \rightarrow \infty} \phi_x(\mathbf{A}_F)^t = 0$. Hence, for any $\varepsilon > 0$, there exists a sufficiently large t_ε such that for $t > t_\varepsilon$,

$$P(t) \leq P_w + \varepsilon \mathbf{1} \quad (7.23)$$

Next, we seek the lower bound evaluation equation for $p(t)$. For all $1 \leq n \leq N$ and all technologies $\gamma(\tau) = n$, we choose a value $1 > \delta > 0$ such that

$$w' l_{(\tau)} \geq \delta p_{wn} . \quad (7.24)$$

More specifically, we can choose δ to be less than the minimum of

$$\min \left(\min_n \left(\min_{\gamma(\tau)=n} \frac{w' l_{(\tau)}}{p_{wn}} \right), 1 \right) . \quad (7.25)$$

Next, choose and fix a $1 > \eta_0 > 0$ such that

$$p(0) \geq \eta_0 p_w \quad (7.26)$$

and define η_t according to the following recurrence relation:

$$\eta_t = \eta_{t-1} + (1 - \eta_{t-1}) \delta, \quad t \geq 1 \quad (7.27)$$

Then, it follows that

$$0 < 1 - \eta_t < 1, \quad (7.28)$$

and

$$\eta_t \nearrow 1, \quad (7.29)$$

along with

$$p(t) \geq \eta_t p_w, \quad (7.30)$$

since (7.27) implies that

$$1 - \eta_t = (1 - \delta)(1 - \eta_{t-1}) = \cdots = (1 - \delta)^t (1 - \eta_0) \quad (7.31)$$

Considering $1 > \eta_0 > 0$, equation (7.28) follows. Moreover, (7.29) is derived from $1 - \delta < 1$, $\eta_t < 1$, and (7.31).

Equation (7.30) can be proven by mathematical induction. It is evident for $t = 0$. Assuming the validity of $t - 1$, demonstrating the validity at t can be achieved by transforming (7.15) as follows.

$$\begin{aligned} P(t)_n &= \min_{\gamma(\tau)=n} (P(t-1)' A_{(\tau)} + w' L_{(\tau)}) \\ &\geq \min_{\gamma(\tau)=n} (\eta_{t-1} P'_w A_{(\tau)} + w' L_{(\tau)}) \quad (\text{by the induction hypothesis}) \\ &= \min_{\gamma(\tau)=n} (\eta_{t-1} (P'_w A_{(\tau)} + w' L_{(\tau)}) + (1 - \eta_{t-1}) w' L_{(\tau)}) \\ &\geq \eta_{t-1} \min_{\gamma(\tau)=n} (P'_w A_{(\tau)} + w' L_{(\tau)}) + (1 - \eta_{t-1}) \delta P_{wn} \quad (\text{choice of } \delta) \\ &= \eta_{t-1} P_{wn} + (1 - \eta_{t-1}) \delta P_{wn} \\ &= (\eta_{t-1} + (1 - \eta_{t-1}) \delta) P_{wn} \\ &= \eta_t P_{wn} \quad (\text{recurrence relation for } \eta_t) \end{aligned} \quad (7.32)$$

The inequality in the 5th line utilizes $\eta_{t-1} > 0$, $1 - \eta_{t-1} > 0$ and (7.24). The sixth equation utilizes the fact that p_w is equal to the minimum cost for each good (Proposition 19).

Therefore, for any $\varepsilon > 0$, by choosing a sufficiently large t , we have $\eta_t \geq 1 - \varepsilon$, and from (7.32),

$$p(t) \geq (1 - \varepsilon) p_w. \quad (7.33)$$

Combining (7.23) and (7.33), we obtain $\lim_{t \rightarrow \infty} p(t) = p_w$. Summarized as a proposition:

【命題 40 Convergence of iterative price adjustment: Starting from any positive initial prices $p(0)$ and positive wage $w \in \hat{W}$ in the iterative price adjustment of (7.15), it converges to $p' = w' \phi_x(\mathbf{L}_F) \phi_x(\mathbf{G}_F)^{-1}$. Here, $\mathbf{G}_F, \mathbf{L}_F$ is the maximal technology face of w 's technology matrix, and x is the productive operating scale of F .

Even if the given wage falls within multiple wage face-positive regions, the price in equation (6.6) is equal by Proposition 36, so there is no need to consider which point in which wage face-positive region to treat as.

If for some reason prices detach from the prices associated with wage w , they will converge back to the original price associated with wage through iterative price adjustment (Proposition 40). In this sense, prices associated with wage w are globally stable.

【命題 41 Global stability of associated prices: Prices associated with wage $w \in \hat{W}$ are globally stable.

We refer to the technique that achieves minimum costs at each price revision through iterative price adjustment as the minimum cost technique. How will the set of minimum cost techniques change as we repeat price adjustments?

Upon evaluating the profit $\pi_{(\tau)}(t) := p'(t) g_{(\tau)} - w' l_{\tau}$ of each technique $c_{(\tau)}$ at given wage w and price $p(t)$ in equation (7.15), the technique that yields a profit of zero is the t -th minimum cost technique, while profits of all other techniques are negative. The profit of technology not belonging to the maximal technology frontier can be expressed as shown in Equation (7.4.20), where it does not depend on t and tends to converge to zero for sufficiently large t . This implies that the profit of technologies not belonging to the maximal technology frontier remains negative after a certain number of revisions and is consistently not adopted.

The cost of the t iteration of technologies belonging to the maximal technology frontier, τ , is not necessarily the minimum cost technology. The first term on the left-hand side of Equation () represents the cost of technology τ , while the second term corresponds to the minimum cost of production using technology τ . If $p(t-1)$ converges to the limit p_w , the cost per utilization of technology τ would be equal to the minimum cost. However, in general, $p(t-1)$ and p_w are not equal, so the cost per utilization of technology τ may not necessarily be equal to the minimum cost. Therefore, not all technologies under the maximal technology face are guaranteed to be adopted for every utilization. Nonetheless, the convergence of the difference between the cost

and the minimum cost of technology τ to zero, as indicated by the limit on the right-hand side of Equation () ($t \rightarrow \infty$), implies that pre-adjustment technologies yield profits equivalent to the markup rate in the limit, suggesting that practically all technologies would be adopted.

【命題 42Convergence to Maximal Technology Face in Practice: The profits of all technologies belonging to the technology face converge to 0, and the profits of pre-adjustment technologies converge to profits calculated at the markup rate. In this sense, all technologies in the technology face are x_{ij} adopted.

7.4 Reasons for Ignoring Values Outside the Value Face

We have not analyzed points technologically feasible outside the technology face, and values outside the value face. Let us explain why it is acceptable to limit the analysis in this way.

Let the price p differ from the associated price of wages w . In that case, according to Proposition 41, prices converge to p_w . Even if prices differ from p_w , such discrepancies are only temporary phenomena, thus it is permissible to ignore them.

8 Changes in Wages, Demand, and Movement of the Technology Face

8.1 Changes in Wages

Suppose wages change from w_I to w_{II} . Let the maximal technology faces be denoted as $F_I := C(w_I)$ and $F_{II} := C(w_{II})$, respectively, distinguished by their technology matrices with subscripts I and II . When the wage changes from w_I to w_{II} , firms engage in iterative price adjustment (7.15) starting from the initial price $p'_I = w'_I \phi_I (\mathbf{L}_I) \phi_I (\mathbf{G}_I)^{-1}$ and w_{II} , and prices converge to $p'_{II} = w'_I \phi_{II} (\mathbf{L}_I) \phi_{II} (\mathbf{G}_{II})^{-1}$. At this point, the technology frontier essentially converges to the maximal technology frontier associated with wage w_{II} .

8.2 Changes in Demand

Given the wage level of $w \in \hat{W}$, the international economy is located at the maximal technology frontier $C(w)$. Prices equal the associated price of wage w . The technology matrix before adjustment is represented by \tilde{C} . If $C(w)$ is productive, then \tilde{C} is also productive, hence Proposition 22 holds for \tilde{C} . For any demand c , \tilde{C} has an operating scale x that realizes the demand, satisfying $\tilde{C}_F x = c$.

As long as $\mathbf{L}x \leq \hat{l}$ holds, firms have no incentive to change prices even when demand c changes. When the dimension of the maximal technology face, denoted as $C(w)$, is less than $(M + N - 1)$, the value face $\mathcal{P}(C(w))$ is not uniquely determined even though it is of at least two dimensions with the scale factor excluded, and there is no reason for prices to change from

the associated price. Changes in the value face are independent of variations in net output quantity or employment within the technology face. Let us summarize this proposition.

【命題 43Independence of Quantity and Prices: In international economics, variations in the quantity of output and employment within the maximal technology face are independent of changes in goods prices, nominal wages, and other factors within the value face.

8.3 Movement to Higher-Dimensional Technology Faces

When there is a change in goods demand, the economy moves within the maximal technology face. A shift to a different technology face does not occur solely due to changes in demand. To transition to a different technology face, not only must there be changes in goods demand, but also shifts in wages, affecting both the minimal wage face and the maximal technology face. Focusing solely on the goods space prevents the observation of differences between various technology faces. Recognition of these differences in technology faces is only possible when considering the entire commodity space, including labor.

Considering the constraints on available labor is a way to reflect the differences in technology faces in the goods space. The factor driving the economy to transition to different technology faces is the constraint on available labor. Let's focus on a subset of the technology face that satisfies the labor input constraint.

【定義 40Attainable operating scale of the technology face is defined as the set of operating scales that satisfy the labor input constraint of the technology face, denoted as the attainable operating scale of the technology face and represented by X_F .

$$X_F := \left\{ x \geq 0 \mid L_F x \leq \hat{l} \right\}. \quad (8.1)$$

When dealing with the net output quantity that meets world demand, it is necessary to use the pre-markup-rate-adjusted \tilde{G}_N as the goods' net output matrix, rather than the post-adjustment G .

It is possible to satisfy world demand when

$$d \leq \tilde{G}_F x, \quad x \in X_F. \quad (8.2)$$

【定義 41Technology Face Reachable Domain: The quantity of goods defined by the attainable operating scale of technology face F and the original technology matrix is referred to as the technology face reachable domain of technology face F , denoted as $\tilde{F}_{(P)}$.

$$\tilde{F}_{(P)} := \left\{ y \in \tilde{C}_F x, x \in X_F \right\}. \quad (8.3)$$

Even if $x \in X_F$, $\tilde{C}_F x$, and $C_F x$ are the same, they are distinct.

The following proposition is trivial.

【命題 44 Inclusion Relation of Reachable Domains: If $F \supset G$, then $\tilde{F}_{(P)} \supset \tilde{G}_{(P)}$.

Initially, consider the state where the economy is positioned at the maximal technology face $G := C(w_I)$ of wage $w_I \in \tilde{W}$'s reachable domain $\tilde{G}_{(P)}$. If demand changes and production in $\tilde{G}_{(P)}$ can no longer meet the quantity of demand. In order for wages to adjust and move to another technology face with a larger feasible space than $\tilde{G}_{(P)}$, wages must change. By moving from point w_I on the minimal wage face $W(w_I) (= P_W(G))$ to point w_{II} on a lower wage face $P_W(F)$, the feasible space expands from $\tilde{G}_{(P)}$ to $\tilde{F}_{(P)}$, enabling the fulfillment of demand (see Figure 1). Importantly, under the assumption that the technologically feasible set C remains unchanged, movement to a different face will eventually lead a country to reach full employment, resulting in an increase in wages, expanding the set of technologies that satisfy the price-cost equality equation, and consequently enabling the ability to meet demand.

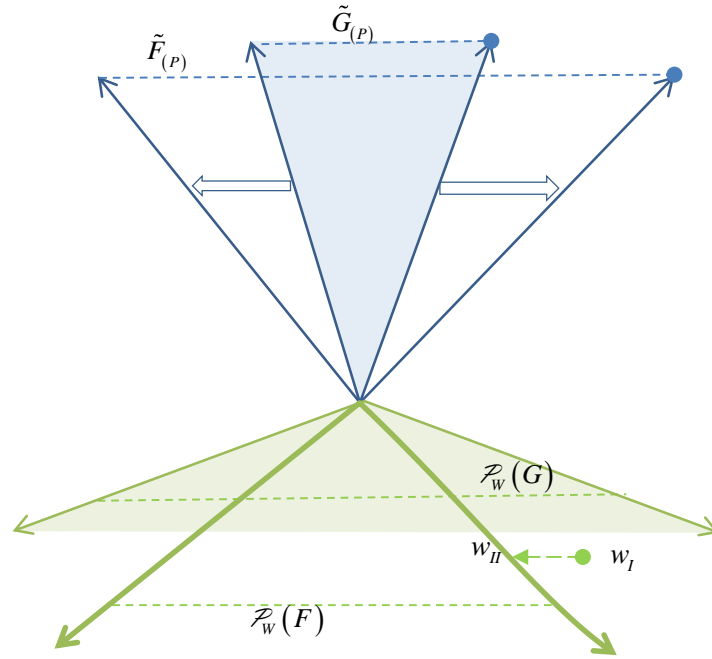


図 1 Movement to a higher-dimensional technology face

where $P_W(G) \supsetneq P_W(F)$. By moving from point $P_W(G) \setminus P_W(F)$ to a narrower point $P_W(F)$, the technology face G expands to a wider F .

8.4 No movement to lower-dimensional technology face

In the previous section, the mechanism for the economy moving to a higher-dimensional technology face was explained. Is it possible for an economy to shift from its current technology

face to a lower-dimensional technology face? The answer is no. This is because the lower-dimensional technology face is already encompassed within the current technology face, so there is no need for the minimal wage face to change even if demand shifts. In contrast, moving to a higher-dimensional technology face would require a shift in wages towards smaller wage faces to accommodate the expansion of the technology face. However, since the low-dimensional technology face is already included in the high-dimensional technology face, there is no reason for wages to change.

Could there be a scenario where an economy transitions from its current technology face to a different technology face that includes a lower-dimensional technology face? Similar to the previous paragraph, this transition cannot occur due to the inability to move from one technology face to a lower-dimensional one contained within it. Therefore, it is reasonable to conclude that no movement to a lower-dimensional technology face takes place.

9 Factors Determining the Upper Limit of Relative Wage

Let's consider the factors that determine the upper limit of the relative wage ratio between countries. A hint lies in the two 1-dimensional wage faces of labor input type for 2 countries and 2 goods $((w_{II}), (w_{III}))$, as shown in Figure 1. The points where these intersect represent the maximum relative productivity ratios (w_A and w_B), which are determined by each country's relative productivity in goods where they have a comparative advantage. Can such a characteristic hold in a general M country's economy with N goods of intermediate goods input type? This point will be further discussed.

Several definitions will be added for clarity.

【定義 42 Wage Simplex and Relative Wage: A $M - 1$ -dimensional simplex that represents wages is referred to as the *wage simplex*. The wages on the wage simplex are termed as *relative wage*.

【定義 43 Wage Extreme Point: The intersection of a wage extreme halfline (defined in Definition 24) with the wage simplex is defined as a *wage extreme point*.

As evident from the definitions, the number of facets on the technologically feasible set C is equal to the number of wage extreme halflines and wage extreme points. This count is denoted by K .

The wage extreme points of eligible facets F can be determined as follows. When F is a facet, r inside Equation (6.40) is equivalent to $M + N - 1$, thereby allowing us to express Equation (6.40) as follows using $r - N = M - 1$. The labor matrix embodied is denoted by \mathbf{E}_{M-1} .

$$\mathbf{W}'\mathbf{E}_{M-1} = 0 \quad (9.1)$$

Matrix \mathbf{E}_{M-1} is of type $M \times (M - 1)$ with rank $M - 1$ (Proposition 26). Since the M row vectors of matrix \mathbf{E}_{M-1} are linearly dependent, Equation (9.1) has a non-zero solution and its solution

space is one-dimensional. Taking the transpose of (9.1) gives $\mathbf{E}'_{M-1}w = 0$, so the solution space is the kernel of the linear transformation $\mathbf{E}'_{M-1}w$. Its dimension is equal to $M - \text{rank}\{\mathbf{E}_{M-1}\}$ (Satake, 1974, Theorem 7 on p. 104). The non-negative region of the solution space forms the wage ray, with wage extreme points being the intersections of the ray and the wage simplex.

Let $\mathbf{E}_{[\setminus m]}$ be the square matrix obtained by removing any m rows of matrix \mathbf{E}_{M-1} . If the technology face is eligible, the rank of matrix $\mathbf{E}_{[\setminus m]}$ is equal to $M - 1$. This is because for positive wages w , Equation (9.1) holds, implying that any row of matrix \mathbf{E}_{M-1} can be expressed as a linear combination of the other rows (Equation 9.3) is an exception to the use of the symbol e as stipulated in the legend). The $e_{[m]}$ represents the m rows of \mathbf{E} . It is also expressed as a linear combination of the row vectors of $\mathbf{E}_{[\setminus m]}$. Assuming that the rank of $\mathbf{E}_{[\setminus m]}$ is less than or equal to $M - 2$, the maximum number of linearly independent row vectors of $\mathbf{E}_{[\setminus m]}$ is also less than or equal to $M - 2$. However, as $e_{[m]}$ is a linear combination of the row vectors of $\mathbf{E}_{[\setminus m]}$, the maximum number of linearly independent row vectors between $e_{[m]}$ and $\mathbf{E}_{[\setminus m]}$ is also less than or equal to $M - 2$. This contradicts the fact that the rank of \mathbf{E}_{M-1} is equal to $M - 1$.

【命題 45 Rank of labor matrix embodied excluding one row: For any m rows of eligible facets of labor matrix embodied by \mathbf{E}_{M-1} , excluding one row at index $(M - 1)$, the rank of the resulting square matrix $\mathbf{E}_{[\setminus m]}$ is equal to $M - 1$.

Let the wage of country m at the k -th ($1 \leq k \leq K$) wage extreme point be denoted by w_{mk} . Define the maximum value of w_{mk} as $w_{m \max} := \max_{1 \leq k \leq K} w_{mk}$.

The $w_{m \max}$ indicates the maximum value of relative wage achievable by country m under the technological conditions of the technology matrix \mathbf{C} in international economics. This is because any point of intersection between a wage face and a wage simplex is a convex combination of wage extreme points, therefore the w_m is clearly less than or equal to $w_{m \max}$.

Let's determine the conditions for the facet of country m with the highest relative wage. Consider the system of equations with respect to the $M - 1$ -dimensional column vector x :

$$\mathbf{E}_{[\setminus m]}x = 1_{M-1} \quad (9.2)$$

From Proposition 45, the solution to (9.2) always exists. Multiplying the solution x to (9.1) from the right, we obtain:

$$0 = w'_{\setminus m} \mathbf{E}_{[\setminus m]}x + w_m e_{[m]}x = w'_{\setminus m} 1_{M-1} + w_m e_{[m]}x. \quad (9.3)$$

Since it holds on the gold simplex, substituting this into (9.3) yields $0 = 1 - w_m + w_m e_{[m]}x$. Solving this, we get:

$$W_m = \frac{1}{1 - e_{[m]}x}. \quad (9.4)$$

The denominator of (9.4) is positive. This is because, starting from (9.3) where $w_m e_{[m]}x = -w'_{\setminus m} 1_{M-1}$, and noting that $w > 0$, we see that $e_{[m]}x$ is negative. The following proposition holds:

【命題 46Transformation by Embodied Labor Matrix: Let an eligible facet F be represented by a labor matrix embodied in \mathbf{E} . In the technology face F , by inputting the labor input of country m as $-e_{[m]}x > 0$, all countries except country m can produce a basket of goods equivalent to 1 embodied good. Here, x is the solution to

$$\mathbf{E}_{[\setminus m]}x = 1_{M-1} \quad (9.5)$$

The proposition in **【Proposition 46】** demonstrates that the technology face can be interpreted as a transformation function from $-e_{[m]}x$ to 1_{M-1} . It is convenient to assign a name to $-e_{[m]}x$.

【定義 44Reference Labor Amount: The $-e_{[m]}x$ within the transformation by embodied labor matrix, as defined in Proposition 46, is referred to as the reference labor amount of country m in the technology face F .

The reference labor amount generally varies within the same country when the technology face changes. Similarly, when the technology face is the same but the country differs, the reference labor amount also generally differs.

From (9.4) and Proposition 46, we have:

【命題 47Facet with Upper Limit of Relative Wages: The facet in country m where the relative wage is maximized is the facet where the reference labor amount of country m is minimized.

The facet where the reference labor amount is minimized can be interpreted as the facet where the labor of country m exhibits the highest productivity. In this sense, the entire technologically feasible set determines the maximum value of relative wages per country. On the other hand, the choice of technology face is determined by the relative wage. There exists a dual relationship between technology and wages in economics, where the question of "Does technology determine wages?" and "Do wages determine technology?" both yield an affirmative response. However, without clarifying the context of these inquiries, it only leads to confusion.

10 Analysis of Wage Dependency Among Nations

10.1 Methodology for Analyzing Wage Dependency Among Nations

10.1.1 Causes of Wage Dependency Among Nations

The technology facet within the realm of international economics is denoted by F . The dependency of wages across all nations is consolidated in the labor matrix embodied by the technology F , as demonstrated in the first equation of Proposition 28. The wage dependency encapsulated by the labor matrix embodied can be categorized into two types. Firstly, when two nations produce the same goods, the wage levels in these nations must align at the level where the production costs of the same goods are equal. Secondly, if goods1 produced by Country A are used as intermediate goods in the production of goods2 in Country B , the wage levels in both countries are interdependent to ensure that the profit from goods1 and goods2 converges

to zero.

10.1.2 Understanding the General-Type Linkages

Divide the set of nation numbers into K non-overlapping groups. The set of nation numbers belonging to the k -th group ($1 \leq k \leq K$) is denoted as M_k , and the set of technology numbers owned by M_k is represented by T_k . Currently, we are considering a r -dimensional technology space, where the set of all T_k is a subset of T .

$$\bigcup_{k=1}^K T_k = T_F, k \neq h \text{ implies } T_k \cap T_h = \emptyset. \quad (10.1)$$

The set of goods numbers corresponding to the technologies owned by the M_k group can be denoted by

$$N_k := \{n \in \gamma(T_k)\} := \{n \mid g_{n(\tau)} \neq 0, \tau \in T_k\}, 1 \leq k \leq K \quad (10.2)$$

In general, the intersection of two different sets of goods numbers is not empty.

【定義 45 General-Type Linkage Group and Division: With respect to Equation (10.2), when Equation (10.3) holds for N_k ,

$$\bigcup_{k=1}^K N_k = N, k \neq h \text{ implies } N_k \cap N_h = \emptyset \quad (10.3)$$

we refer to each M_k and $1 \leq k \leq K$ as a general-type linkage group, and every country is divided into K general-type linkage groups.

The use of "general-type" in linkage is to distinguish it from the "Graham-type linkage" to be introduced later.

If there exists a general-type linkage among nations as in Equation (10.3), the technology matrix of size F can be represented as in Equation (10.4) by appropriately permuting country indices, goods indices, and technology indices. Conversely, if the technology matrix can be represented as in Equation (10.4), then there exists a general-type linkage like that in Equation (10.3). These facts are evident.

$$\begin{bmatrix} \mathbf{U}_N \mathbf{G}_F \mathbf{U}_T' \\ \mathbf{U}_M \mathbf{L}_F \mathbf{U}_T' \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1 & & & \\ N_1 \times T_1 & & & \\ & \ddots & & \\ & & \mathbf{G}_K & \\ & & N_K \times T_K & \\ \hline & \mathbf{L}_1 & & \\ M_1 \times T_1 & & & \\ & & \ddots & \\ & & & \mathbf{L}_K \\ & & & M_K \times T_K \end{bmatrix} \quad (10.4)$$

Also, in the above equation, \mathbf{U}_M and \mathbf{U}_N denote permutation matrices that respectively rearrange the country and goods indices, while \mathbf{U}_T' represents the column permutation matrix

that rearranges the technology indices. In the subsequent discussion, empty spaces within the matrices are to be interpreted as zeros.

Let us now present a method for understanding the general-type linkage.

【命題 48 Necessary and Sufficient Conditions for Division into General-Type Linkage Group】 The necessary and sufficient condition for dividing M countries into K general-type linkage groups, in the sense of Definition 45, is that the following equation holds by a row permutation matrix \mathbf{U} :

$$\mathbf{U} (\mathbf{L}_F | \mathbf{G}_F |') (\mathbf{U} (\mathbf{L}_F | \mathbf{G}_F |'))' = \begin{bmatrix} \mathbf{M}_1 & & \\ & \ddots & \\ & & \mathbf{M}_K \end{bmatrix}. \quad (10.5)$$

Here, $| \cdot |$ is a matrix with elements as absolute values, \mathbf{U} is a row permutation matrix, and *permutation matrix* holds true as mentioned in Takashiro (1961) page 83. The matrix \mathbf{M}_k is a square matrix of order M_k .

Proof: Necessity. We prove by induction. First, we prove the case when $K = 2$. We can reorder the country numbers so that the number of M_1 is smaller than M_2 (arranged to come first). Similarly, we can reorder the goods numbers so that the number of N_1 is smaller than N_2 , and reorder the technology number so that the number of T_1 is smaller than T_2 . We represent the row permutation matrix for reordering country numbers as \mathbf{U}_M , the row permutation matrix for reordering goods numbers as \mathbf{U}_N , and the column permutation matrix for reordering technology numbers as \mathbf{U}_T' . The technology matrix after renumbering takes the following form. Using equation (), the calculation of $(\mathbf{U}_M \mathbf{L}_F \mathbf{U}_T') (\mathbf{U}_N | \mathbf{G}_F | \mathbf{U}_T')'$ yields:

$$(\mathbf{U}_M \mathbf{L}_F \mathbf{U}_T') (\mathbf{U}_N | \mathbf{G}_F | \mathbf{U}_T')' \quad (10.6)$$

$$= \begin{bmatrix} \mathbf{L}_1 & \\ & \mathbf{L}_2 \end{bmatrix} \begin{bmatrix} | \mathbf{G}_1 |' \\ | \mathbf{G}_2 |' \end{bmatrix} \quad (10.7)$$

$$= \begin{bmatrix} \mathbf{L}_1 | \mathbf{G}_1 |' & \\ & \mathbf{L}_2 | \mathbf{G}_2 |' \end{bmatrix}. \quad (10.8)$$

Therefore, expanding the left-hand side using the property that the product of row permutation matrix and column permutation matrix is an identity matrix:

$$\begin{aligned} & \left((\mathbf{U}_M \mathbf{L}_F \mathbf{U}_T') (\mathbf{U}_N | \mathbf{G}_F | \mathbf{U}_T')' \right) \left((\mathbf{U}_M \mathbf{L}_F \mathbf{U}_T') (\mathbf{U}_N | \mathbf{G}_F | \mathbf{U}_T')' \right)' \\ &= (\mathbf{U}_M \mathbf{L}_F \mathbf{U}_T' \mathbf{U}_T' | \mathbf{G}_F | \mathbf{U}_N) (\mathbf{U}_M \mathbf{L}_F \mathbf{U}_T' \mathbf{U}_T | \mathbf{G}_F | \mathbf{U}_N)' \\ &= \mathbf{U}_M \mathbf{L}_F | \mathbf{G}_F | \mathbf{U}_N \mathbf{U}_N' | \mathbf{G}_F |' \mathbf{L}_F' \mathbf{U}_M' \quad (\text{using } \mathbf{U}_T \mathbf{U}_T' = \mathbf{I}) \\ &= \mathbf{U}_M \mathbf{L}_F | \mathbf{G}_F | | \mathbf{G}_F |' \mathbf{L}_F' \mathbf{U}_M' \quad (\text{using } \mathbf{U}_N \mathbf{U}_N' = \mathbf{I}) \\ &= \mathbf{U}_M \mathbf{L}_F | \mathbf{G}_F | (\mathbf{U}_M \mathbf{L}_F | \mathbf{G}_F |')'. \end{aligned} \quad (10.9)$$

Substituting (10.9) into the left-hand side of (), we obtain

$$\mathbf{U}_M \mathbf{L}_F | \mathbf{G}_F | (\mathbf{U}_M \mathbf{L}_F | \mathbf{G}_F |)' = \begin{bmatrix} (\mathbf{L}_1 | \mathbf{G}_1 |') (\mathbf{L}_1 | \mathbf{G}_1 |')' & \\ & (\mathbf{L}_2 | \mathbf{G}_2 |') (\mathbf{L}_2 | \mathbf{G}_2 |')' \end{bmatrix} \quad (10.10)$$

which resembles the form of the case when $K = 2$. Next, we demonstrate that K holds under the assumption that $K - 1$ holds. This can be expressed in the following form based on the premise. The Japanese economic research paper written in TeX:

$$\begin{bmatrix} \mathbf{U}_N \mathbf{G}_F \mathbf{U}_T' \\ \mathbf{U}_M \mathbf{L}_F \mathbf{U}_T' \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{(1:K-1)} & \mathbf{G}_K \\ \mathbf{L}_{(1:K-1)} & \mathbf{L}_K \end{bmatrix}. \quad (10.11)$$

Where,

$$\mathbf{G}_{(1:K-1)} := \begin{bmatrix} \mathbf{G}_1 & & \\ & \ddots & \\ & & \mathbf{G}_{K-1} \end{bmatrix}, \quad (10.12)$$

$$\mathbf{L}_{(1:K-1)} := \begin{bmatrix} \mathbf{L}_1 & & \\ & \ddots & \\ & & \mathbf{L}_{K-1} \end{bmatrix}. \quad (10.13)$$

Applying the same reasoning as in the case of $K = 2$ to (10.11), we obtain the following.

$$\begin{aligned} & \mathbf{U}_M (\mathbf{L}_F | \mathbf{G}_F |) (\mathbf{L}_F | \mathbf{G}_F |)' \mathbf{U}_M' \\ &= \begin{bmatrix} (\mathbf{L}_{(1:K-1)} | \mathbf{G}_{(1:K-1)} |') (\mathbf{L}_{(1:K-1)} | \mathbf{G}_{(1:K-1)} |')' & \\ & (\mathbf{L}_K | \mathbf{G}_K |') (\mathbf{L}_K | \mathbf{G}_K |')' \end{bmatrix}. \end{aligned} \quad (10.14)$$

By the induction hypothesis, the first block on the right-hand side of (10.14) can be decomposed into $K - 1$ diagonal blocks. This verifies the case when K .

Sufficiency. We prove by induction for the case of $K = 2$. Based on the assumptions,

$$\mathbf{U} (\mathbf{L}_F | \mathbf{G}_F |') (\mathbf{U} (\mathbf{L}_F | \mathbf{G}_F |'))' = \begin{bmatrix} \mathbf{M}_1 & \\ & \mathbf{M}_2 \end{bmatrix} \quad (10.15)$$

can be expressed.

Denote it by $\mathbf{U} (\mathbf{L}_F | \mathbf{G}_F |') =: \mathbf{H}$. \mathbf{H} is a $M \times N$ matrix, where the elements $h_{m \ n}$ represent whether the technology belonging to country m outputs or inputs goods n ; 1 for yes and 0 for no (i.e., if they are unrelated). It is evident that rows represent countries, and columns represent goods.

Split \mathbf{H} . The economic paper written in Japanese using tex is translated into English:

$$U(L_F | G_F |') = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_3 & \mathbf{H}_4 \end{bmatrix}. \quad (10.16)$$

The division of rows consists of the first M_1 rows and the remaining $M - M_1$ rows. The division of columns is arbitrary. Substituting (10.16) into the left side of (10.15), (10.15) can be transformed as follows. The following Japanese economics paper written in LaTeX is translated into English:

$$\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_3 & \mathbf{H}_4 \end{bmatrix} \begin{bmatrix} \mathbf{H}_1' & \mathbf{H}_3' \\ \mathbf{H}_2' & \mathbf{H}_4' \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 \mathbf{H}_1' + \mathbf{H}_2 \mathbf{H}_2' & \mathbf{H}_1 \mathbf{H}_3' + \mathbf{H}_2 \mathbf{H}_4' \\ \mathbf{H}_3 \mathbf{H}_1' + \mathbf{H}_4 \mathbf{H}_2' & \mathbf{H}_3 \mathbf{H}_3' + \mathbf{H}_4 \mathbf{H}_4' \end{bmatrix} \quad (10.17)$$

$$= \begin{bmatrix} \mathbf{M}_1 & \\ & \mathbf{M}_2 \end{bmatrix}. \quad (10.18)$$

Therefore, it must be $\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} \mathbf{H}_3 & \mathbf{H}_4 \end{bmatrix}' = \mathbf{H}_1 \mathbf{H}_3' + \mathbf{H}_2 \mathbf{H}_4' = 0$. However,

$$\begin{aligned} & \text{element } (m_1, m_2) \text{ of } \mathbf{H}_1 \mathbf{H}_3' \\ &= m_1\text{-th row of } \mathbf{H}_1 \times (m_2\text{-th row of } \mathbf{H}_3)' \\ &= \begin{cases} \text{positive, if country } m_1 \text{ and } m_2 \text{ have common input-output goods,} \\ \text{zero, otherwise.} \end{cases} \end{aligned} \quad (10.19)$$

This also holds for $\mathbf{H}_2 \mathbf{H}_4'$. Therefore, given $\mathbf{H}_1 \mathbf{H}_3' \geq 0$ and $\mathbf{H}_2 \mathbf{H}_4' \geq 0$, $\mathbf{H}_1 \mathbf{H}_3' + \mathbf{H}_2 \mathbf{H}_4' = 0$ implies that $\mathbf{H}_1 \mathbf{H}_3' = 0$ and $\mathbf{H}_2 \mathbf{H}_4' = 0$. From equation (10.19), $\mathbf{H}_1 \mathbf{H}_3' = 0$ indicates that there are no goods in common between the countries in group 1 (after country number change) and group 2 that are input-output by them. The same conclusion is drawn from $\mathbf{H}_2 \mathbf{H}_4' = 0$. Thus, M countries are divided into 2 general-type linkage groups in the sense of Definition 45.

Next, we demonstrate that K holds assuming $K - 1$. By rewriting in the form below from the assumption of the inductive method,

$$U(L_F | G_F |') (U(L_F | G_F |'))' = \begin{bmatrix} \mathbf{M}_{(1:K-1)} & \\ & \mathbf{M}_K \end{bmatrix} \quad (10.20)$$

$$\mathbf{M}_{(1:K-1)} := \begin{bmatrix} \mathbf{M}_1 & & \\ & \ddots & \\ & & \mathbf{M}_{K-1} \end{bmatrix} \quad (10.21)$$

Applying the same argument as in the case of $K = 2$ to (10.21), it can be proven that there are no common input-output goods between the countries belonging to Group 1 and those belonging to Group K . Furthermore, based on the assumption of the inductive method, it can be shown that there are no common input-output goods among the countries in Group 1 to Group $K - 1$. Consequently, there are no common input-output goods among all groups. Thus, the sufficient condition has been established. (End of proof)

The terminology for classifying general-type linkages is defined.

【定義 46 Complete General-Type Linkage, Independent General-Type Linkage, Incomplete General-Type Linkage, Non-Complete General-Type Linkage: A linkage without two or more general-type linkage groups is referred to as a Complete General-Type Linkage, a linkage with only one country in each linkage group is referred to as an Independent General-Type Linkage, and a linkage that does not fall into either category is referred to as an Incomplete General-Type Linkage. The combination of Independent General-Type Linkage and Incomplete General-Type Linkage is collectively referred to as a Non-Complete General-Type Linkage.

The number of groups in an Independent General-Type Linkage is equal to the number of countries M .

10.2 Implications of General-Type Linkage on the Wage Face

Assuming that the eligible technology face F is divided into K groups ($K \geq 2$) of general-type linkages, without loss of generality, the technology matrix \mathbf{F} can be expressed as follows. The elements corresponding to \mathbf{G}_k and \mathbf{L}_k in p and q are represented by $p_{(k)}$ and $w_{(k)}$, respectively (Refer to Equation). This notation is used solely in this section for the purpose of symbol efficiency. Note that these elements do not pertain to the k elements of p or w . The conditions of price-cost equality can be expressed as:

$$\begin{bmatrix} p'_{(k)} & w_{(k)'} \end{bmatrix} \begin{bmatrix} \mathbf{G}_k \\ -\mathbf{L}_k \end{bmatrix} = 0, \quad 1 \leq k \leq K. \quad (10.22)$$

The rank of the coefficient matrix on the left-hand side of Equation (10.22) is denoted by r_k . It is evident from the form of Equation (10.22) that $w_{(k)}$ is uniquely determined independently from other $w_{(h)}$, $h \neq k$.

【定義 47 Partial Wage Set: The set obtained by extracting the coordinate portion in wage face $\mathcal{P}_W(F)$ of $w_{(k)}$ is denoted by $W_k(F)$ and referred to as the "partial wage set."

$W_k(F)$ exists in a M_k -dimensional space. Upon conducting a derivation similar to Proposition 31 for (10.22), it follows that $\dim\{W_k(F)\} = M_k + N_k - r_k$ holds true. The wage face $\mathcal{P}_W(F)$ is a direct product set of mutually independent sets $W_k(F)$, $1 \leq k \leq K$, in a M_k -dimensional space.

$$\mathcal{P}_W(F) = \prod_{k=1}^K W_k(F) \quad (10.23)$$

Since the dimension of the left-hand side, denoted as $M + N - \dim\{F\}$, is equal to the dimension of the right-hand side, denoted as $\sum_{k=1}^K (M_k + N_k - r_k)$, the ranks of 2 matrix \mathbf{F} and $\sum_{k=1}^K r_k$ are equal. Therefore,

【命題 49Influence of general-type linkage on wage face: When the technology face F is divided into K general-type linkage groups, the wage face $\mathcal{P}_W(F)$ can be expressed as the direct product set of K independent partial wage sets. Here, "independent" means that each partial wage set is not influenced by other partial wage sets. Moreover, denoting the set of goods numbers and country numbers of the general-type linkage group k ; $1 \leq k \leq K$; as M_k and N_k , and representing the rank of the technology matrix as r_k , the partial wage sets form a polyhedral cone of dimension $M_k + N_k - r_k$.

【命題 50Upper Limit on the Number of General-Type Linkage Groups and the Dimension of Eligible Technology Face:

1. The upper limit on the number of general-type linkage groups of dimension r eligible technology face is $M + N - r$.
2. Considering K as the number of general-type linkage groups, the upper limit on the dimension of the eligible technology face is $M + N - K$.

10.3 The Number of General Linkage Groups and the Dimension of the Technology Face

If the technology face F is eligible, there exists a positive region within the wage face, thus $d\{W_k(F)\} \geq 1$. Summing over all k , we have $d\{\mathcal{P}_W(F)\} = d\left\{\prod_{k=1}^K W_k(F)\right\} = \sum_{k=1}^K d\{W_k(F)\} \geq K$, therefore,

$$d\{\mathcal{P}_W(F)\} = M + N - r \geq K \quad (10.24)$$

From (10.24), we can determine the upper limit on the number of general linkage groups and the dimension of the technology face.

【命題 51】 The Dimension of Eligible Technology Faces and Upper Limit on the Number of General Type Linkage Groups:

1. The upper limit on the number of general type linkage groups for an r -dimensional eligible technology face is $M + N - r$.
2. If K is the number of general type linkage groups, then the upper limit on the dimension r of the eligible technology face is $M + N - K$.

【命題 52The Relationship between Facet and Complete General-Type Linkage - Part 1 (Sufficient Condition of Complete General-Type Linkage): The dimension of a facet is $M + N - 1$, so the number of general-type linkage groups for facets, K , is equal to 1.

However, in general, the reverse is not true. Even if an eligible technology face is a complete general-type linkage, it may not be a facet. The logical relationship between a general-type

complete linkage and the uniqueness of value is as follows: wages are uniquely determined except for scale \Leftrightarrow a technology face is a facet \Rightarrow a technology face is a complete general-type linkage \Rightarrow wage faces are not divided into mutually independent partial wage sets.

An example is provided to demonstrate that a general-type complete linkage is not a sufficient condition for facets. This illustration negates the equivalence between complete general-type linkage and facets. Traditional theories have long focused on complete general-type linkage or the subsequent complete Graham-type linkage as the primary determinants of global wages, but it is now evident that these arguments lacked a solid foundation.

Utilizing Tables 1 through 5, we illustrate a 2-country 3-goods intermediate goods input economy with complete general-type linkage, characterized by a 3-dimensional technology facet rather than a facet of 4 dimensions.

There are a total of 5 technologies (Table 1). Country 1 possesses technologies 1, 2, and 4, each producing goods 1, 2, and 1 respectively. On the other hand, country 2 owns technologies 3 and 5, both exclusively dedicated to the production of good 3.

表4 Numerical Example of a Technology Matrix with a Complete General-Type Linkage but not a Facet Technology Face

	Technology Number	1	2	3	4	5
Technology Matrix↓	Goods Number	1	2	3	1	3
<i>G</i>	Goods 1	1	-0.8	0	1	0
	Goods 2	-0.8	1	-0.5	-0.5	0
	Goods 3	0	-0.5	1	0	1
<i>-L</i>	Labor 1	-1	-0.5	0	-6	0
	Labor 2	0	0	-1	0	-10

Technology numbers 1, 2, 3, and 4 constitute an eligible facet (a 4-dimensional technology face), as evaluated in Table 5. Profits for technology numbers 1, 2, 3, and 4 are all zero, while technology 5 incurs a loss. This indicates that technology numbers 1, 2, 3, and 4 form the frame of a 4-dimensional technology face and, given the positive value of q_1 , confirms the eligibility of this technology face.

表 5 Quadrant 1 of 4-dimensional technology face

	Quadrant 1 of 4-dimensional technology face				Outside technology face	
Technology number	1	2	3	4	5	
Goods number	1	2	3	1	3	Value
Goods1	1	-0.8	0	1	0	p_1 13.
Goods2	-0.8	1	-0.5	-0.5	0	p_2 15
Goods3	0	-0.5	1	0	1	p_3 8.8
Labor1	-1	-0.5	0	-6	0	w_1 0.9
Labor2	0	0	-1	0	-10	w_2
Profit evaluated with q_1	0	0	0	0	-1.1875	

Technology numbers 1, 2, 3, and 5 form distinct eligible facets in a separate 4-dimensional technology face (see Table 6).The benefits of the technology evaluated by q_2 are zero for technologies 1, 2, 3, and 5, and negative for technology 4. This confirms that technologies 1, 2, 3, and 4 constitute the four-dimensional technology face frame. Furthermore, since the value of q_2 is positive, it can be confirmed that this technology face is eligible.Inserting table 3:

表 6 Four-dimensional Technology Face 2

	Four-dimensional Technology Face 2				Outside Technology Face	
Technology Number	1	2	3	5	4	
Goods Number	1	2	3	3	1	Value of q_2
Goods 1	1	-0.8	0	0	1	p_1 15.5385
Goods 2	-0.8	1	-0.5	0	-0.5	p_2 18
Goods 3	0	-0.5	1	1	0	p_3 10
Labor 1	-1	-0.5	0	0	-6	w_1 1.1385
Labor 2	0	0	-1	-10	0	w_2 1
Profits Evaluated by q_2	0	0	0	0	-0.2923	

The three-dimensional technology face formed by technologies 1, 2, and 3 at the intersection of these two is shown in Table 7.In this paper written in LaTeX, $\mathbf{G}_{(1:3)}$ and $\mathbf{L}_{(1:3)}$ represent the net output matrix and labor input matrix, respectively, of the 3-dimensional technology space. The value cone formed by q_1 and q_2 constitutes a polyhedral cone, within which the profits for technologies 1, 2, and 3 are zero for any arbitrary value. Furthermore, the profits for technologies 4 and 5 are negative for values within the relative interior of the value cone.Insert

Table 4:

表 7 3-Dimensional Technology Face

		3-Dimensional Technology Face		
Technology Matrix	Technology Number	1	2	3
	Goods Number	1	2	3
$\mathbf{G}_{(1:3)}$	Goods 1	1	-0.8	0
	Goods 2	-0.8	1	-0.5
	Goods 3	0	-0.5	1
$-\mathbf{L}_{(1:3)}$	Labor 1	-1	-0.5	0
	Labor 2	0	0	-1

It can be confirmed that this 3-Dimensional Technology Face is a complete general-type linkage by observing the forms of matrices $\mathbf{L}_{(1:3)}\mathbf{G}_{(1:3)}'$ (Table 8) and $\mathbf{L}_{(1:3)}\mathbf{G}_{(1:3)}'(\mathbf{L}_{(1:3)}\mathbf{G}_{(1:3)}')'$ (Table 9).

表 8 $\mathbf{L}_{(1:3)}\mathbf{G}_{(1:3)}'$

	Goods1	Goods2	Goods3
Labor 1	-0.6	0.3	0.25
Labor 2	0	0.5	-1

表 9 $\mathbf{L}_{(1:3)}\mathbf{G}_{(1:3)}'(\mathbf{L}_{(1:3)}\mathbf{G}_{(1:3)}')'$

	Labor 1	Labor 2
Labor 1	0.5125	-0.1
Labor 2	-0.1	1.25

In light of the significant findings, these are summarized into a proposition.

【命題 53 Relationship between Facet and Complete General-Type Linkage - Part 2 (Counterexample): An eligible complete general-type linkage's technology facet is not necessarily a facet. In other words, the value is not uniquely determined even when excluding scale.

10.4 Graham-Type Linkage

Graham analyzed the dependency between nations in the international economy that produce goods by inputting only labor. If country 1 and country 2 both produce goods 1, then they are directly linked in the Graham's sense. If country 2 and country 3 both produce goods 2, then not only are they directly linked in the Graham's sense, but country 1 and country 3 are indirectly linked in the Graham's sense through country 2. Let's call this concept of linkage Graham-Type Linkage. Graham-Type Linkage is a concept of linkage based solely on the identity of goods produced by countries. In contrast, the concept of General-Type Linkage from Definition 45 considers not only the identity of goods produced by countries but also the interconnections used in the production of other countries' goods.

Graham-Type Linkage is a concept of linkage that uses matrix \mathbf{B} as a technology matrix instead of \mathbf{G} in Definition 45. Specifically, the following equation is used instead of (10.2):

$$N_k^{(G)} := \{n \in \gamma(T_k)\} := \{n \mid b_{n(\tau)} \neq 0, \tau \in T_k\}, \quad 1 \leq k \leq N. \quad (10.25)$$

【定義 48Graham-Type Linkage Group and Division: For the $N_k^{(G)}$ defined by (10.25), if

$$\bigcup_{k=1}^K N_k^{(G)} = N, \quad k \neq h \text{ implies } N_k^{(G)} \cap N_h^{(G)} = \emptyset \quad (10.26)$$

then, each $N_k^{(G)}$ and $1 \leq k \leq K^{(G)}$ are referred to as a *Graham-Type Linkage Group*, and it is stated that all countries are *divided* into $K^{(G)}$ Graham-Type Linkage Groups.

The proof of the following proposition is omitted as it is similar to that of Proposition 48.

【命題 54Necessary and Sufficient Condition for Division into a Graham-Type Linkage Group: The necessary and sufficient condition for dividing M countries into $K^{(G)}$ Graham-type linkage groups, as defined in Definition 48, is that there exists a row permutation matrix \mathbf{U} such that Equation (10.27) holds:

$$\mathbf{U} (\mathbf{L}_F \mathbf{B}') (\mathbf{U} (\mathbf{L}_F \mathbf{B}'))' = \begin{bmatrix} \mathbf{M}_1^{(G)} & & \\ & \ddots & \\ & & \mathbf{M}_{K^{(G)}}^{(G)} \end{bmatrix} \quad (10.27)$$

Even if two countries belong to a common general-type linkage group, they may not belong to a common Graham-type linkage group. However, if two countries belong to the same Graham-type linkage group, they also belong to the same general-type linkage group.

【命題 55Relation between Graham-type Linkage and General-type Linkage: If two countries $k \neq h$ belong to the same Graham-type linkage group, then they belong to the same general-type linkage group.

Proof: To prove the contrapositive proposition that "If two countries do not belong to the same general-type linkage group, then they do not belong to the same Graham-type linkage group." We will only prove the case of $K = 2$, but the general case can be proven in a similar manner. The technology matrix can be represented as follows by permuting country numbers, goods numbers, and technology numbers. The Japanese economic paper written in LaTeX is translated into English as follows:

$$\begin{bmatrix} \mathbf{U}_N \mathbf{G}_F \mathbf{U}_T' \\ \bar{\mathbf{U}}_M \bar{\mathbf{L}}_F \bar{\mathbf{U}}_T' \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 \\ \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{bmatrix} \begin{matrix} N_1 \times T_1 \\ N_2 \times T_2 \\ M_1 \times T_1 \\ M_2 \times T_2 \end{matrix}. \quad (10.28)$$

Applying the same permutation to \mathbf{B}_F , we obtain the following.

$$\begin{bmatrix} \mathbf{U}_N \mathbf{B}_F \mathbf{U}_T' \\ \bar{\mathbf{U}}_M \bar{\mathbf{L}}_F \bar{\mathbf{U}}_T' \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \\ \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{bmatrix} \begin{matrix} N_1 \times T_1 \\ N_2 \times T_2 \\ M_1 \times T_1 \\ M_2 \times T_2 \end{matrix}. \quad (10.29)$$

If $g_n(\tau) = 0$ holds true, it implies that technology τ does not produce goods n , leading to $b_n(\tau) = 0$. This establishes that when two countries belong to different general-type linkage groups, they belong to distinct Graham-type linkage groups. (End of Proof)

The terminology for classifying Graham-type linkages is defined.

【定義 49 Complete Graham-type linkage, complete specialization type, incomplete Graham-type linkage: The state where there are no more than two Graham-type linkage groups is referred to as complete Graham-type linkage, the state where each country within a Graham-type linkage group is only one is referred to as complete specialization type, and the state that does not fall into either category is referred to as incomplete Graham-type linkage.

The number of Graham-type linkage groups in the complete specialization type and the number of countries M are equal.

For $\mathbf{A} = 0$, Graham-type linkage aligns with general-type linkage. Hence, similar propositions to the one in **【Proposition 52】** hold true for Graham-type linkage as well.

【命題 56 Sufficient Conditions for Complete Graham-Type Linkage: If $\mathbf{A} = 0$, and the eligible technology face is a facet, then it is a complete Graham-type linkage.

10.5 Summary of General States of Linkage

In a general intermediate goods input economy, even if the eligible technology face is a facet, it may not be a complete Graham-type linkage. Moreover, there are cases where it is a complete general-type linkage but simultaneously becomes a complete specialization type. This occurs when all countries produce different goods and are linked to each other through intermediate goods input-output relationships.

Figure 2 illustrates the most common dependency in international economics. There can be multiple general-type linkage groups. Within one general-type linkage group, there can be multiple Graham-type linkage groups, and within a Graham-type linkage group, there can be one or more countries. From this figure, it is evident that Graham-type linkage groups alone cannot fully represent the dependency among nations.

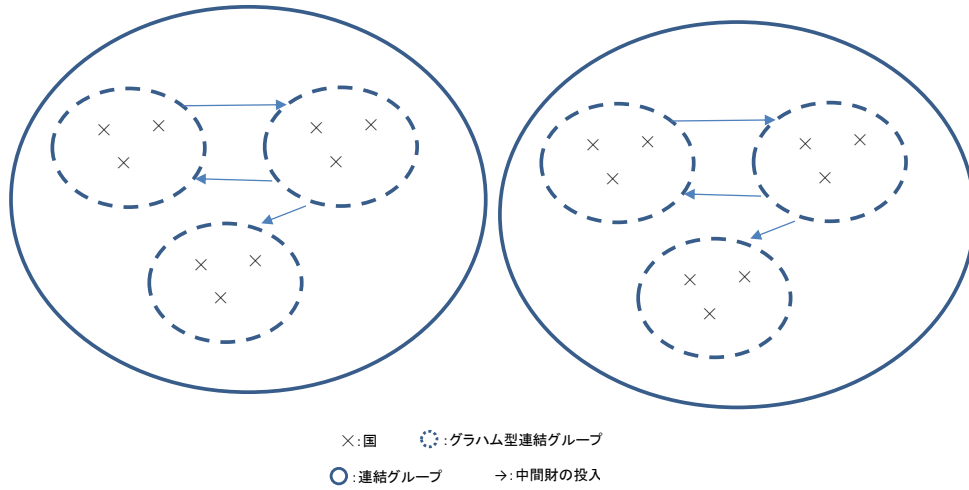


図2 General State of Linkage

10.6 Creation of Complete Specialization Type Technology Face

10.6.1 Method of Creating Complete Specialization Type Technology Face in Any Dimension

The purpose of this subsection is to demonstrate the lack of relationship between the state of complete specialization and the dimension of the technology face. Assuming $N > M$, the dimension of the technology face is represented as r rather than $N + r$. Starting from N dimensions, it is shown that it is possible to construct eligible technology face of complete specialization type up to $M + N - 1$ dimensions. Table (10) illustrates the completed form of a technology matrix in $N + r$ dimensions, which can aid in understanding the proof. Insertion of Table 10.5.1:

表 10 Technology Matrix of a complete specialization type in arbitrary dimensions

		Technology Number											
		1	, ...	, ...	, r	, ...	, ...	, M	, ... n ...	, N		, N + 1	, ... , N
Goods	1	1	0	, ...	, ...	, ...	, ...	0	...	0		1	0
Number	\vdots	0	\ddots	\ddots	\ddots	\ddots	\ddots	\vdots	\ddots	\vdots		$-g_2$	\ddots
	\vdots	\vdots	\ddots	\ddots	\ddots	\ddots	\ddots	\vdots	\ddots	\vdots		\ddots	\ddots
	r	0	\ddots	\ddots	1	\ddots	\ddots	\vdots	\ddots	0		0	\ddots
	r + 1	\vdots	\ddots	\ddots	0	\ddots	\ddots	\vdots	\ddots	\vdots		0	\ddots
	\vdots	\vdots	\ddots	\ddots	\vdots	\ddots	\ddots	\vdots	\ddots	\vdots		\vdots	\ddots
	M	\vdots	\ddots	\ddots	\vdots	\ddots	\ddots	1	\ddots	\vdots		0	\ddots
	\vdots	\vdots	\ddots	\ddots	\vdots	\ddots	\ddots	0	\ddots	0		\vdots	\ddots
	N	0	0	...	1		0	0
Country	1	-1	0	0				$-l_1$	0
Number	\vdots	\vdots	\ddots	\ddots	\vdots	\ddots	\ddots	\vdots				0	\ddots
	\vdots	\vdots	\ddots	\ddots	\vdots	\ddots	\ddots	\vdots				\vdots	\ddots
	r	0	-1	0	$\cdots -l_{(n)} \cdots -l_{(N)}$			\vdots	\ddots
	\vdots	\vdots	\ddots	\ddots	\vdots	\ddots	\ddots	\vdots				\vdots	\ddots
	M	0	0	-1				0	0

Consider a technology matrix $\mathbf{C}_{(N)}$ consisting of technologies $\mathbf{B}_{N \times N} = \mathbf{I}_{N \times N}$, $\mathbf{A}_{N \times N} = 0$ and $\mathbf{L}_{M \times N} \geq 0$. If $1 \leq \tau \leq M$, then assume $m \neq \gamma(\tau)$ and $l_{m(\tau)} = 0^{*10}$. The numbering of $\mathbf{B} = \mathbf{I}$ implies that the first M technologies correspond to the same goods numbers. The condition regarding labor input is that technologies τ from the 1st to the M th are assigned to country τ . This condition holds even after rearranging the country numbers if needed. Since each country produces only one type of goods in $\mathbf{C}_{(N)}$, it is evident that $\mathbf{C}_{(N)}$ is a technology matrix of the complete specialization type, and its rank is N , as stated in $\mathbf{B}_{N \times N} = \mathbf{I}_{N \times N}$.

Assume validity at $r - 1$ implies validity at r : Add the technology $c_{(N+r)}$ determined by the following equation to the known technology matrix $\mathbf{C}_{(N+r-1)}$ to construct $\mathbf{C}_{(N+r)} :=$

^{*10} By choosing the unit to measure labor input in each country, it is possible to reconcile $b_{\tau\tau} = 1$ and $l_{\tau\tau} = 1$.

$$\begin{bmatrix} \mathbf{C}_{(N+r-1)} & c_{(N+r)} \end{bmatrix}.$$

$$\mathbf{C}'_{(N+r)} = (\mathbf{G}'_{(N+r)} \quad \mathbf{L}'_{(N+r)}) \quad (10.30)$$

$$\mathbf{G}'_{(N+r)} = \begin{pmatrix} 0 & \cdots & 0 & \underbrace{1}_r & \underbrace{-g_{r+1}}_{r+1} & 0 & \cdots & 0 \end{pmatrix}, \quad G_{r+1} > 0 \quad (10.31)$$

$$\mathbf{L}'_{(N+r)} = \begin{pmatrix} 0 & \cdots & 0 & \underbrace{-l_r}_r & 0 & \cdots & 0 \end{pmatrix}, \quad L_r > 0 \quad (10.32)$$

Technology $c_{(N+r)}$ belongs to country r , using only goods $r + 1$ as intermediate inputs to produce goods r .

Let's demonstrate the inconsistency that arises if we assume that technology $c_{(N+r)}$ can be represented as a linear combination of $\mathbf{C}_{(N+r-1)}$. There exists a pair of coefficients, represented by α_i , $1 \leq i \leq N + r - 1$, such that

$$\mathbf{G}_{(N+r)} = \sum_{j=1}^{N+r-1} \alpha_j \mathbf{G}_{(j)}, \quad (10.33)$$

$$\mathbf{L}_{(N+r)} = \sum_{j=1}^{N+r-1} \alpha_j \mathbf{L}_{(j)} \quad (10.34)$$

are assumed to hold. Focusing on the $r + 1$ row of Equation (10.33), we observe that the columns with non-zero elements are limited to column $1 \leq j \leq N$, with a value of 1. In the range of $N + 1 \leq j \leq N + r$, only column $N + r$ is non-zero, with a value of $-g_{r+1}$. Consequently,

$$-G_{r+1} = \alpha_{r+1}. \quad (10.35)$$

Rows beyond row $g_{(N+1)}$, $g_{(N+2)}$, \dots , $g_{(N+r)}$ are all 0, with row $g_{(j)}$, $1 \leq j \leq N$, being the only row with a value of 1, while all other rows are 0. Therefore, for $r + 1$ or more rows of the labor matrix and up to i rows, we focus on the j column. In the range $1 \leq j \leq M$, only the i row is -1 while the other rows are 0. In the range of $M + 1 \leq j \leq N$, due to the assumption $r \leq M - 1$, we have $r + 2 \leq j \leq N$ from (), which implies $\alpha_j = 0$. For the range of $N + 1 \leq j \leq N + r$, the j column is 0. Hence, for $r + 1 \leq i \leq M$,

$$\begin{aligned} 0 &= \sum_{\substack{1 \leq k \leq M \\ k \neq i}} \alpha_k \cdot 0 + \alpha_i + \sum_{M \leq j \leq N} \alpha_j [-l_{ij}] + \sum_{N+1 \leq j \leq N+r-1} \alpha_j \cdot 0 \\ &= \alpha_i. \end{aligned} \quad (10.36)$$

The second equality utilizes (10.36). However, $g_{r+1} > 0$ holds because of equation (10.35), implying $\alpha_{r+1} \neq 0$. On the other hand, from equation (10.36), we have $\alpha_{r+1} = 0$ and α_{r+1} , which are contradictory conditions. Hence, $c_{(N+r)}$ and $\mathbf{C}_{(N+r-1)}$ are linearly independent. It is important to note that linear independence holds for any $g_{r+1} > 0$, $l_r > 0$.

Furthermore, the country producing goods r with technology $c_{(N+r)}$ is r country. Since, by the assumption of induction, $\mathbf{C}_{(N+r-1)}$ is a technology matrix of the complete specialization type,

the only country capable of producing goods r using technology $\mathbf{C}_{(N+r-1)}$ is also r country. Consequently, the country producing goods r with the technology matrix $\mathbf{C}_{(N+r)}$ is only r country. By a revision of the induction, it can be shown that each country produces other goods in a single country. Thus, it is proven that the technology matrix $\mathbf{C}_{(N+r)}$ is a technology matrix of the $N + r$ -dimensional complete specialization type.

Next, we aim to demonstrate that the technology face formed by this constructed technology matrix is eligible. To show that technology face (\mathbf{C}_{M+N-1}) is eligible, it suffices to prove the eligibility of the contained technology face $(\mathbf{C}_{N+r}), 0 \leq r \leq N - 2$, which is evident. Therefore, the focus is on proving eligibility for \mathbf{C}_{M+N-1} . From the price-cost equality condition,

$$[P' \quad W'] [\mathbf{C}_{(N)} \quad c_{(N+1)} \quad \cdots \quad c_{(M+N-1)}] = 0. \quad (10.37)$$

The rank of the coefficient matrix in the system of equations (10.37), with q unknowns, is $M + N - 1$, hence it is evident that a unique solution exists if we determine w_1 . Let us seek specific solutions to establish the necessary and sufficient conditions for the existence of positive q . By solving for the first M equations in (10.37) and from the $(N+1)$ -th to the $M + N - 1$ -th equation, the wages for M and the prices for goods 1 to goods M are determined. The first M equations imply

$$1 \leq n \leq M \text{ such that } p_n = w_n. \quad (10.38)$$

And the equations from $(N + 1)$ to $(M + N - 1)$ yield

$$p_r = p_{r+1}g_{r+1} + w_rl_r, \quad 1 \leq r \leq M - 1. \quad (10.39)$$

Substituting (10.38) into (10.39) gives $w_r = w_{r+1}g_{r+1} + w_rl_r, \quad 1 \leq r \leq M - 1$, which can be transformed as follows:

$$\begin{aligned} w_{r+1} &= w_r \frac{(1 - l_r)}{g_{r+1}}, \quad 1 \leq r \leq M - 1, \\ &= w_1 \prod_{k=1}^r \frac{(1 - l_k)}{g_{k+1}}, \quad 1 \leq r \leq M - 1. \end{aligned} \quad (10.40)$$

Starting from w_1 , (10.40) leads to a recurrence relation determined up to w_M . Choosing $l_r < 1$ so that w_1 are positive, all goods prices w_2, \dots, w_M are positive after substituting them into (10.38). The remaining goods prices p_{M+1}, \dots, p_M come from the equations from the $(M+1)$ th to the N th. The relationship between an N -dimensional technology face and a complete specialization type is derived from Equation (), which states that if wages are positive, then goods prices are also positive. These values, crucial for later use, are denoted as q_I .

The quantities are defined as follows:

$$q_I' = [p_I' \quad w_I'], \quad (10.41)$$

$$w_I' = \left(w_1, \dots, w_1 \prod_{k=1}^{r-1} \frac{(1 - l_k)}{g_{k+1}}, \dots, w_1 \prod_{k=1}^{M-1} \frac{(1 - l_k)}{g_{k+1}} \right), \quad (10.42)$$

$$p_I' = (\dots, w_{\chi(n)}l_{\chi(n)}, \dots), \text{ where } 1 \leq n \leq M \text{ and } \chi(n) = n. \quad (10.43)$$

These positive wages and prices satisfy the price-cost equality equation at all points on the technology face, demonstrating the eligibility of the technology face.

10.7 N -Dimensional Technology Face and the Relationship with Complete Specialization Type

【命題 57 N -dimensional Technology Face and Complete Specialization Type: Consider a N -dimensional technology face that produces all goods^{*11}.

1. In a labor-input economy, a N -dimensional technology face consists of complete specialization type and is composed of N frame technologies (implying the column number of the technology matrix is N).
2. In an intermediate goods input economy, a N -dimensional technology face could be of complete specialization type or not.

Proof of 1: Let N -dimensional technology face be represented by the frame matrix F . Given the assumption of $G_F = B_F$, and normalizing the output, F is obtained. Let the set of numbers corresponding to F linearly independent columns be denoted by T_F . Within T_F , there exists a technology to produce any goods. Assuming that there is no technology to produce goods 1 within T_F , the first row of $B_{(T_F)}$ would be all zeros, contradicting the assumption that there exists technology to produce any goods within the set of F linearly independent columns, including goods 1. Thus, there exists a basis vector to produce any goods n . By relabeling the technology numbers, we can have $T_F = \{1, \dots, N\}$, and further relabeling the goods numbers, we can have $n = \gamma(n), 1 \leq n \leq N$. At this point, without loss of generality, we can consider technology 1 to N as the basis vectors. After this relabeling, $B_{(T_F)}$ becomes a N -dimensional identity matrix denoted as I_N .

Next, selecting and fixing an arbitrary goods number n , any technology to produce goods n can be expressed as:

$$\begin{bmatrix} b_{(h)} \\ -l_{(h)} \end{bmatrix} = \sum_{1 \leq \tau \leq N} \alpha_\tau \begin{bmatrix} b_{(\tau)} \\ -l_{(\tau)} \end{bmatrix} \quad (10.44)$$

Proof of statement 110: For goods other than n , the left-hand side is zero, and all terms on the right-hand side except for i are zero, which implies i . Hence, any technology that produces goods n is a scalar multiple of the basic technology that produces the same goods, revealing that it is the basic technology itself. This fact indicates that the technology face in a N -dimensional labor-input economy is of the complete specialization type.

Proof of Part 2: Table 11 provides an example of a 3-dimensional technology face in a 2-country, 3-goods model, which is not of the complete specialization type based on intermediate goods input. There are three technologies where technology 1 and technology 2 produce goods

^{*11} This proposition holds even when the technology matrix is not productive.

1, while technology 3 produces goods 2 (as indicated by the output matrix \mathbf{B}_F). The country of affiliation for technology 1 is Country 1, and for technology 2 and technology 3, it is Country 2 (according to the labor matrix \mathbf{L}_F), meaning both Country 1 and Country 2 produce goods 1.

The determinant of the 3-dimensional square matrix \mathbf{G}_F is non-zero, which implies that the rank of \mathbf{G}_F is 3. Therefore, the rank of this technology face is at least 3 and at most the number of columns, which is 3, making it equal to 3. Hence, the dimension of this non-complete specialization type technology face is 3.

表 11 Numerical Example of Non-complete Specialization Type with Technology Face
Where Dimensions Equal Number of Goods

	Technology Number	1	2	3		
Technology Matrix	Goods Number	1	1	2		
\mathbf{B}_F	Goods 1	1	1	0		
	Goods 2	0	0	0		
	Goods 3	0	0	1		
\mathbf{A}_F	Goods 1	0.37	0.41	0.02		
	Goods 2	0.27	0.94	0.58		
	Goods 3	0.52	0.05	0.55	Value	
\mathbf{G}_F	Goods 1	0.63	0.59	-0.02	p_1	19.68
	Goods 2	-0.27	-0.94	-0.58	p_2	10.41
	Goods 3	-0.52	-0.05	0.45	p_3	16.51
\mathbf{L}_F	Labor 1	1	0	0	w_1	1
	Labor 2	0	1	1	w_2	1
Technological Profit		0	0	0		
Determinant of \mathbf{G}_F		-0.026				

Table 12 presents an example of a 3-dimensional technology face with complete specialization type involving 2 countries and 3 goods in the intermediate goods input scenario. In the economic paper written in Japanese using tex, it is stated that there are three technologies that produce different goods (based on the output matrix \mathbf{B}_F). The country of affiliation for technology 1 is Country 1, while the country of affiliation for technologies 2 and 3 is Country 2 (as indicated by the labor matrix \mathbf{L}_F). Consequently, Country 1 and Country 2 exhibit a complete specialization type.

Since the determinant of the net output matrix \mathbf{G}_F is non-zero, the rank of \mathbf{G}_F is equal to 3. Therefore, the rank of this technology face is at least 3 and at most 3, implying that the

dimension of this complete specialization type technology face is 3.

表 12 Numeric Example of the Number of Complete Specialization Types with Technology Face Equal to the Number of Goods in Dimensions

	Technology Number	1	2	3		
Technology Matrix	Goods Number	1	2	3		
\mathbf{B}_F	Goods 1	1	0	0		
	Goods 2	0	1	0		
	Goods 3	0	0	1		
\mathbf{A}_F	Goods 1	0.02	0.51	0.15		
	Goods 2	0.09	0.22	0.56		
	Goods 3	0.05	0.09	0.64	Value	
\mathbf{G}_F	Goods 1	0.98	-0.51	-0.15	p1	1.8
	Goods 2	-0.09	0.78	-0.56	p2	3.49
	Goods 3	-0.05	-0.09	0.36	p3	8.96
\mathbf{L}_F	Labor 1	1	0	0	w1	1
	Labor 2	0	1	1	w2	1
Profits of Technology		0	0	0		
Determinant of \mathbf{G}_F		0.188				

10.8 The Relationship between Two Types of Complete Linkage and Facets

It has been proven so far that the following.

1. If a facet, then it is a complete general-type linkage (Proposition). The converse is not necessarily true (Table 7 provides an example).
2. If it is a complete Graham-type linkage, then it is a complete general-type linkage (Proposition 55).

For the case of $\mathbf{A}_F = 0$, if it is a complete Graham-type linkage, then it becomes a facet. However, it is currently uncertain if the same holds for $\mathbf{A}_F \neq 0$. Figure 3 illustrates what is known so far.

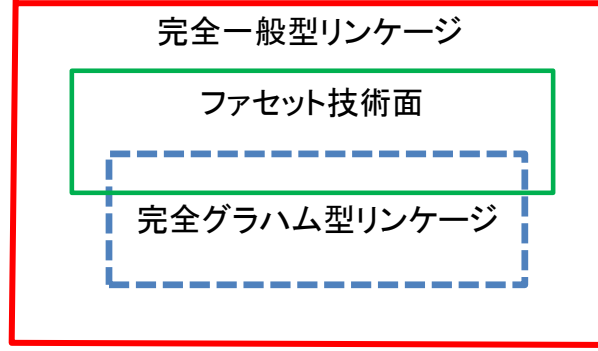


図3 The relationship between types of complete linkage and the facet

11 Counterexample to the "Dichotomy" of Labor-Input Economy with 2 Countries and 2 Goods

11.1 The "Dichotomy" of Labor-Input Economy with 2 Countries and 2 Goods

The labor-input economy with 2 countries and 2 goods has distinct characteristics concerning types of linkage, the dimension of the value face, and pairings such as gains from trade.

1. The minimum dimension of the eligible technology face that produces all goods is 2, with the accompanying dimension of the value face also being 2. As the dimension of the value face is 2, prices and wages are considered to move freely, and values are determined such that the demand and supply quantities of commodities (goods and labor) coincide. Gains from trade occur in both countries on this technology face, leading each country to specialize in the production of goods in which they have a comparative advantage.
2. The maximum dimension of the eligible technology face that produces all goods is 3, which corresponds to the facet. The dimension of the value face associated with the facet is 1. In economics, the dimension of the value face being one-dimensional results in a unique determination of value apart from scale. This state is referred to as fixed pricing in contrast to the two-dimensional value face. The linkage type of this facet is a complete Graham-type linkage, where all countries compete in the final goods market. This is considered to be the reason why value is uniquely determined apart from scale.

The purpose of this section is to present a counterexample showing that the dichotomous characteristics of a 2-country, 2-goods labor-input economy may not hold in a more general M country N goods intermediate goods input economy.

11.2 Numerical Example of a Facet that is Complete Specialization Type

An example of a facet that is complete specialization type but a facet is shown (Counterexample 1 of Section 11.1). Table 13 provides a numerical example of a 3-country, 6-goods intermediate goods input economy with a facet that is complete specialization type (an 8-dimensional technology face).

Country 1, 2, and 3 each possess technologies to produce goods numbered (1, 4, 7), (2, 5, 8), and (3, 6, 9) respectively. As each country produces different goods, they exhibit complete specialization type.

At least two facets exist in Table 13. Facet 1 consists of technology numbers (1, 2, 3, 4, 5, 6, 7, 8), while facet 2 comprises technology numbers (1, 2, 3, 4, 5, 6, 7, 9). Each facet holds a positive value, and when evaluated at that value, the technologies belonging to the facet have a profit of zero, while those not belonging have a negative profit. As the entire technology matrix is of complete specialization type, the two facets are also of complete specialization type.

表 13 Numerical Example of the Technology Facet that is of the complete specialization type despite being a facet

Technology No.	1	2	3	4	5	6	7	8
Goods No.	1	2	3	4	5	6	1	2
Goods1	1	0	0	0	0	0	1	0
Goods2	0	1	0	0	0	0	-0.8	1
Goods3	0	0	1	0	0	0	0	-0.8
Goods4	0	0	0	1	0	0	0	0
Goods5	0	0	0	0	1	0	0	0
Goods6	0	0	0	0	0	1	0	0
Labor1	-1	0	0	-0.5	0	0	-0.5	0
Labor2	0	-1	0	0	-0.5	0	0	-0.8
Labor3	0	0	-1	0	0	-0.5	0	0

Facet1	120	120	120	120	120	120	120	120
Value	198	160	100	128	80	50	198	160
	2.56	1.6	1	1.28	0.8	0.5	2.56	1.6
Technology No.	1	2	3	4	5	6	7	8
Technology Profit	0	0	0	0	0	0	0	0

The Technology No. 120 (Complete Specialization) indicates whether the technology is included in the facet

Facet2	Year1928	Year1928	Year1928	Year1928	Year1928	Year1928	Year1928	Year1928
Value	1.25	, 0.7813	, 1	, 0.625	, 0.3906	, 0.5	, 1.25	, 0.7813
Technology Number	1	, 2	, 3	, 4	, 5	, 6	, 7	, 8
Technology Profit	0	, 0	, 0	, 0	, 0	, 0	, 0	, 0

11.3 Example of a facet that is not a complete Graham-type linkage

We present an example of a facet that is not a complete Graham-type linkage but still constitutes a facet (see 11.1.2 for a counterexample). Table 14 provides numerical data for a 2-country, 3-goods intermediate goods input economy with a facet that is not a complete Graham-type linkage. In an economic study written in Japanese using TeX, the polyhedral cone defined by technologies 1, 2, 3, and 4 has positive value, making it a technology face. Due to its dimension

being 4, it is considered a facet.

表 14 Numerical Example of a technology face that is facet but not fully Graham-type

Technology ID	1	2	3	4	5	6	7
Goods ID	1	2	3	1	1	2	3
Goods1	1	-0.8	0	1	1	-0.9	0
Goods2	-0.8	1	-0.5	-0.5	-0.9	1	0
Goods3	0	-0.5	1	0	0	0	1
Labor1	-1	-0.5	0	-6	-0.9	-5	0
Labor2	0	0	-1	0	0	0	-10

Facet	○	○	○	○	×	×	×
Value	p1	p2	p3	w1	w2		
	13.4375	15.625	8.8125	0.9375	1		
Technology ID	1	2	3	4	5	6	7
Technology Profit	0	0	0	0	-1.4688	-1.1562	-1.1875

Figure illustrates the state of this facet's Graham-type linkage. h denotes the physical characteristics number and m_c represents the consumer country number, identifying a pair of goods (h, m_c) .

By assigning numbers to all pairs of goods (h, m_c) , the set of goods numbers is denoted as $\{1, 2, \dots, N\}$.

The mapping from the set of technology numbers T to the set of goods numbers N represents the goods produced by technology τ . Equivalently, it can be described as the mapping from the set of technology numbers T to the set of pairs of two goods $\{(1, 1), (1, 2), \dots, (1, M), (2, 1), \dots, (H, M)\}$ as $\gamma(\tau) = (h(\tau), m_c(\tau))$.

With the increased variety of goods, adjustments are made to the technology vector. The output vector $b_{(\tau)}$ of technology τ has elements where only $b_{\gamma(\tau)(\tau)}$ are positive, while the other elements are zero (Assumption 1, item 1). The production activities of technology τ are carried out in the country represented by $\chi(\tau)$, so the actual consumption country attribute of inputted goods is only $\chi(\tau)$. Therefore, the input vector of technology τ , $a_{(\tau)}$, is zero unless the consumption country attribute of input goods is $\chi(\tau)$ ($m_c \neq \chi(\tau)$ implies $a_{(h, m_c)(\tau)} = 0$).

The transport from country m_p to country m_c is considered as a production process that involves inputting goods from the origin, packaging materials, and transport services (such as air or truck transport), in order to produce goods at the destination. When labor from

multiple countries is used in transportation, the entire process can be divided into the number of relevant countries, with each subprocess inputting labor from one country and modeling the production of goods to be used in the next subprocess. It can be represented as a series of processes where goods are processed sequentially like $(h, m_0) = (h, m_p) \rightarrow (h_T, m_1) \rightarrow \dots \rightarrow (h_T, m_{L-1}) \rightarrow (h, m_L) = (h, m_c)$. Here, h_T is a physical characteristics number representing goods in transit^{*12}.

The set of technology numbers is represented by $T := \{1, 2, \dots, T\}$.

Tax rates are dependent on the consumption country and physical characteristics of goods. In the case of discriminatory tariffs, they also depend on the producing country. Given that the producing country is determined by technology number τ , tax rates can be identified by goods number and technology number, represented as $\beta_{\gamma(\tau), \tau}$. Firms may adjust the markup rate in response to discriminatory tariffs, so the markup rate is also identified by the goods number and the technology number, denoted as $\alpha_{\gamma(\tau), \tau}$ ^{*13}.

We explain the iterative price adjustment process for taxed prices (see Figure). Given the wage w , the price $p(t-1)$ of the $t-1$ -th iteration of goods ($t \geq 1$), the initial technology matrix, markup rate, and tax rate, the price $p_{\gamma(\tau)}(t)$ of the t -th iteration of goods $\gamma(\tau)$ with technology τ is determined.

$$p_{\gamma(\tau)}(t) b_{\gamma(\tau), \tau} = (1 + \alpha_{\gamma(\tau), \tau} + \beta_{\gamma(\tau), \tau}) (p'(t-1) a_{(\tau)} + w_{\chi(\tau)} l_{\chi(\tau)}). \quad (11.1)$$

The price $p_{(h, m_c)}(t)$ of the t -th iteration of goods (h, m_c) is determined by the principle of minimum pricing. The vector consisting of $p_{(h, m_c)}(t)$ is $p(t)$. Replace t with $t-1$ and return to (11.1). This process repeats iteratively. The initial value of goods prices, $p(0) > 0$, is arbitrary.

When we divide both sides of (11.1) by $(1 + \alpha_{\gamma(\tau)} + \beta_{\gamma(\tau)})$, we obtain the following equation.

$$p_{\tau} \frac{b_{\gamma(\tau), \tau}}{1 + \alpha_{\gamma(\tau), \tau} + \beta_{\gamma(\tau), \tau}} = p'(t-1) a_{(\tau)} + w_{\chi(\tau)} l_{\chi(\tau)}. \quad (11.2)$$

The output of modified technology $(1 + \alpha_{\gamma(\tau)} + \beta_{\gamma(\tau)})^{-1}$ after revising the matrix defined in Equation 11.3,

$$g_{(\tau)} = (1 + \alpha_{\gamma(\tau), \tau} + \beta_{\gamma(\tau), \tau})^{-1} b_{(\tau)} - a_{(\tau)}. \quad (11.3)$$

can be redefined as the modified technology matrix \mathbf{C} , which allows for the analysis under a zero tariff rate to be directly applicable.

^{*12} Goods with physical characteristics h and consumed in country m as (h, m) are different goods from those with physical characteristics h but located in country m during transport as (h_T, m) .

^{*13} Once the markup rate τ is determined, the production country $\chi(\tau)$ is also determined, leading to the determination of the markup rates and tax rates for each production country.

11.4 Export Restrictions

In some cases, the export of strategic goods to a specific country may be restricted for political reasons. By appropriately modifying the current model, it can be applied to an international economy where export restrictions exist.

When the export of goods with physical characteristic \bar{h} to a specific country \bar{m} is restricted, adjustments should be made in the treatment of technology $\bar{\tau}$ associated with that particular country. It involves excluding the technology to produce $\gamma(\tau) = (\bar{h}, \bar{m})$ from countries that impose restrictions (e.g., the United States) and from other countries that align with the export restriction policies. The technology to produce $\gamma(\tau) = (\bar{h}, \bar{m})$ from foreign technology is not excluded. This enables the modeling of export restrictions.

11.5 Import Restrictions

Import restrictions can also be handled similarly to export restrictions. When country m imposes restrictions on the import of goods with physical characteristics \bar{h} from country \bar{m} , except for the technology to produce $\gamma(\tau) = (\bar{h}, m)$ from country \bar{m} , import restrictions can be modeled in the same framework.

12 Gains from Trade

12.1 Conditions for the Existence of Gains from Trade

We examine the benefits of initiating trade (referred to as gains from trade) for a country in a closed economy (referred to as the target country).

There are two methods to analyze gains from trade. The first method examines whether the set of goods available for consumption in the target country expands before and after trade initiation. This method is limited by the assumption of full employment of resources and cannot address unemployment. The second method compares the real wage of the target country before and after trade initiation. This method does not require the assumption of full employment. However, this method also does not address the impact of trade on employment in the target country, which is a separate issue. In this section, we explore the existence of gains from trade using the second method.

Without loss of generality, we can refer to the target country as Country 1 (the country number can be changed if necessary). We compare the real wage of Country 1 before and after trade initiation. The input-output matrix used by Country 1 before trade initiation is denoted as \mathbf{A}_C , \mathbf{B}_C , and the labor vector is denoted as l_C (in a closed economy). Since we analyze labor for a single country, we extract Country 1's row from the labor matrix \mathbf{L} . The column numbers of \mathbf{A}_C , \mathbf{B}_C , and l_C are denoted by T_C . For the sake of symbol economy, T_C also represents the

set of pre-trade technology numbers adopted by country 1. The technology matrix is normalized so that the positive elements of \mathbf{B}_C are equal to 1.

From Proposition 27, specifically from 2, the value of country 1 pre-trade is uniquely determined. Country 1 is located at a productive N -dimensional eligible technology face before trade begins, and there exists a productive pair $(x_C, \mathbf{B}_C - \mathbf{A}_C)$. The technology-goods mapping corresponding to pair $(x_C, \mathbf{B}_C - \mathbf{A}_C)$ is denoted by ϕ_{x_C} and succinctly represented by ϕ_C .

Country 1's wage is used as the value scale. The pre-trade goods price of country 1 is denoted by p_C , and the post-trade goods price is denoted by p_O (opened economy). Both are positive. The net benefit of country 1's pre-trade adopted technology, evaluated at p_C , is equal to zero. The image of the technology-goods mapping according to Equation (13.1.1) is as follows:

$$P_c'(\mathbf{I} - \phi_c(\mathbf{A}_c)) = \phi_c(L_c). \quad (12.1)$$

Some technologies of Equation 2050, Equation 2051, and Equation 2048 become inoperative and lose cost competitiveness after trade initiation.

【定義 50Inferior technology: The technologies that were operational before trade initiation but become inoperative afterward are referred to as inferior technology.

The set of numbers for inferior technology is denoted by Equation 2068 (inferior). The set of numbers for non-inferior technology among Equation 2069 is represented by Equation 2070 (survival). Naturally, Equation 2071 holds. Evaluating the adopted technologies before trade initiation with Equation 2062, the profit of inferior technology is negative, while the profit of non-inferior technology is equal to zero.

$$\text{For } \tau \in T_C^{(I)}, p_O' (b_{C(\tau)} - a_{C(\tau)}) < l_C \tau. \quad (12.2)$$

$$\text{For } \tau \in T_C^{(S)}, p_O' (b_{C(\tau)} - a_{C(\tau)}) = l_C \tau. \quad (12.3)$$

【定義 51Inferior Technology's Product or Inferior Goods: Goods produced by inferior technology before the start of trade are referred to as inferior technology's product, or simply inferior goods.

The set of numbers for inferior goods is denoted by $N^{(I)}$, and for others by $N^{(S)}$. Clearly, $N = N^{(I)} \cup N^{(S)}$, $N^{(I)} \cap N^{(S)} = \emptyset$. Without loss of generality, it may be assumed that $N^{(I)} = \{1, \dots, N_1\}$ and $N^{(S)} = \{N_1 + 1, \dots, N\}$.

Consider the image under the mapping determined by ().

$$\text{For } n \in N^{(I)}, p_O'(\mathbf{I} - \phi_C(\mathbf{A}_C))_{(n)} < \phi_C(l_C)_n. \quad (12.4)$$

The profit of non-inferior technology producing inferior goods equals zero. However, when combined with the negative profit of inferior technology producing the same goods through weighted averaging, the average profit of producing inferior goods becomes negative. On the one hand,

$$\text{for } n \in N^{(S)}, p_O'(\mathbf{I} - \phi_C(\mathbf{A}_C))_{(n)} = \phi_C(l_C)_n. \quad (12.5)$$

From (12.1)(12.4), and (12.5),

$$\text{for } n \in N^{(I)}, (p'_O - p'_C) (I - \phi_C (\mathbf{A}_C))_{(n)} < 0, \quad (12.6)$$

$$\text{for } n \in N^{(S)}, (p'_O - p'_C) (I - \phi_C (\mathbf{A}_C))_{(n)} = 0. \quad (12.7)$$

The price of inferior goods decreases as shown in (12.6). Furthermore, even if the goods are not inferior goods, if they use inferior goods as intermediate goods, their prices will also decrease. This can be demonstrated as follows. The price change of goods n is expressed as follows:

$$\begin{aligned} (p'_O - p'_C)_n &= \left((p'_O - p'_C) (I - \phi_C (\mathbf{A}_C)) (I - \phi_C (\mathbf{A}_C))^{-1} \right)_n \\ &= (p'_O - p'_C) (I - \phi_C (\mathbf{A}_C)) (I - \phi_C (\mathbf{A}_C))_{(n)}^{-1} \\ &= \sum_{i \in N^{(I)}} \overbrace{((p'_O - p'_C) (I - \phi_C (\mathbf{A}_C)))_{(i)}}^{1 \times N} (I - \phi_C (\mathbf{A}_C))_{[i](n)}^{-1} \\ &\quad + \sum_{i \in N^{(S)}} \overbrace{((p'_O - p'_C) (I - \phi_C (\mathbf{A}_C)))_{(i)}}^{1 \times N} (I - \phi_C (\mathbf{A}_C))_{[i](n)}^{-1} \\ &= \sum_{i \in N^{(I)}} (p'_O - p'_C) (I - \phi_C (\mathbf{A}_C))_{(i)} (I - \phi_C (\mathbf{A}_C))_{[i](n)}^{-1} \\ &\quad + \sum_{i \in N^{(S)}} (p'_O - p'_C) (I - \phi_C (\mathbf{A}_C))_{(i)} (I - \phi_C (\mathbf{A}_C))_{[i](n)}^{-1} \\ &= \sum_{i \in N^{(I)}} (p'_O - p'_C) (I - \phi_C (\mathbf{A}_C))_{(i)} (I - \phi_C (\mathbf{A}_C))_{[i](n)}^{-1}. \end{aligned} \quad (12.8)$$

The equivalence in the first and fourth equalities exploits that the product of the $\mathbf{X}\mathbf{Y}$ matrix and the n column of prices matches the product of columns \mathbf{X} and \mathbf{Y} of prices. From the last equation, the fact that the second term of the fifth side is 0, as derived from equation (13.1.8), was utilized. If goods n are input as intermediate goods as inferior goods, the term of those intermediate goods will be negative in the final side. This is because the product of the first factor and the second factor is negative, as indicated by equation (13.1.7), and the third factor is positive as it is directly or indirectly input as intermediate goods. Therefore, the overall $(p'_O - p'_C)_n$ will also be negative.

【定義 52 Generalized inferior technology's product or generalized inferior goods: Goods that are produced using inferior goods themselves and inferior goods used as inputs directly or indirectly are referred to as the generalized inferior technology's product, or simply as generalized inferior goods.

Figure 4 illustrates the relationship among inferior technology, inferior goods, and generalized inferior goods. The interpretation is as follows: view the inferior technology vertically. Technologies that operate after trade and produce goods identical to those of inferior technology are classified as non-inferior technology. Technologies that process goods produced by inferior tech-

nology, the goods processed by those technologies, and so forth, are categorized as generalized inferior goods.

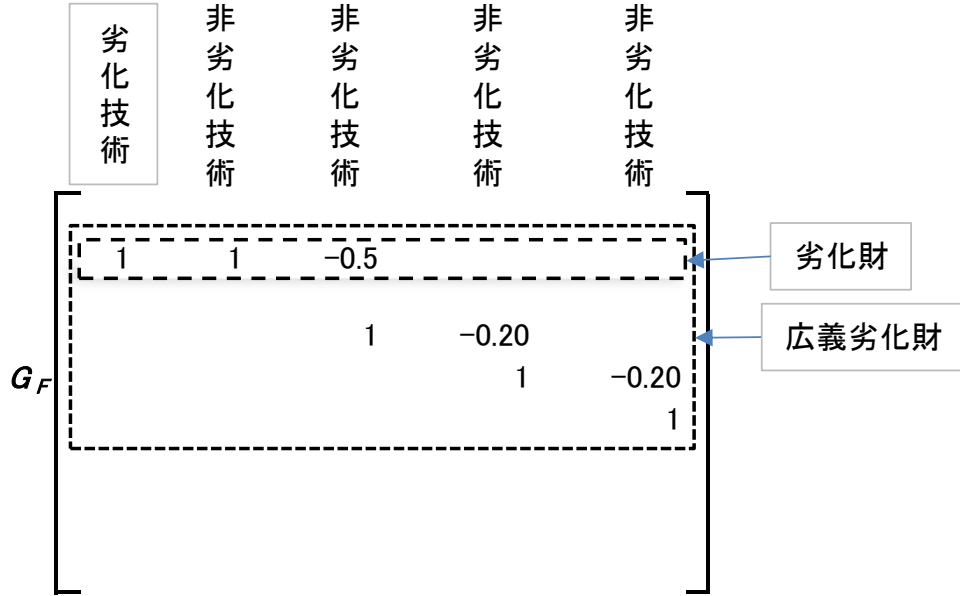


図 4 Inferior Goods and Generalized Inferior Goods

【定義 53Country 1's Gains from Trade: Country 1 has gains from trade if the prices of all goods after trade are higher than before trade, but at least one good has a lower price.

From (12.8), we obtain the following proposition.

【命題 58Equivalent Conditions for Country 1's Gains from Trade: The following conditions are equivalent for Country 1:

1. Country 1 has gains from trade.
2. Country 1 has inferior technology.
3. Country 1 has generalized inferior goods.

Moreover, from (12.8),

【命題 59Price Changes Before and After Trade:

1. The price of Country 1's generalized inferior goods decreases.
2. The real wage measured in terms of the price of generalized inferior goods rises for Country 1.
3. The real wage measured in terms of prices other than generalized inferior goods remains unchanged.
4. After the initiation of trade, the real wage does not decrease regardless of the goods prices

used for measurement.

The proposition following is directly derived from Proposition 59.

【命題 60Changes in Real Wages for Country 1:

1. The price of generalized inferior goods as measured real wage increases.
2. The real wage measured in terms of prices other than generalized inferior goods remains the same.
3. There is no decrease in the real wage when measured with any goods price after the initiation of trade.

Following the commencement of trade, even if Country 1 produces goods domestically, the prices of those goods decrease due to the effect of lower prices for imported inferior goods used as intermediate goods. This cost reduction effect spreads widely through the input-output structure, resulting in lower prices for many goods.

The analysis of gains from trade through changes in real wage was first conducted by Ikema (1991). Ikema examined a labor-input economy of Country 3 goods. The conclusion was that "the real wage rises when measured by imported goods prices, but remains constant when measured by goods continuing domestic production". However, Proposition 60 reveals that the latter is not accurate in the presence of intermediate goods. If domestic products were completely replaced by imports, it is evident that the goods are inferior goods. The real wage measured by the price of imports would increase. However, even if imports and domestic production coexist without complete substitution by imports, if the technologies adopted before the start of trade are no longer employed, then the goods are inferior goods, and the goods that directly or indirectly input these goods for production are generalized inferior goods. Generalized inferior goods constitute a much broader category of goods than imported goods.

As a result of Proposition 60, we obtain Proposition 61.

【命題 61Sufficient Conditions for Gains from Trade for All Countries: The following are sufficient conditions for gains from trade for all countries:

1. If every country has at least one inferior technology, the real wage measured by the price of inferior goods will increase. Consequently, gains from trade occur in all countries.
2. If every country has at least one inferior technology, the real wage measured by the price of generalized inferior goods will increase. This implies gains from trade for all countries.
3. If every country has at least one inferior technology, the real wage measured by the price of the basket of all goods will increase. In this sense, gains from trade occur in all countries.

12.2 Numerical example for gains from trade to occur in all countries on the facet

First, a notable characteristic of a labor-input economy is that the number of technologies a country possesses is *effectively* equal to the number of goods. This is because, even if Country 1 has multiple technologies to produce Good 1, the technology with the smallest labor input will always be chosen, rendering the existence of other technologies irrelevant. This is due to the fact that only the domestic labor costs affect the production cost, while other factors (such as the prices of other goods and the wages in other countries) do not. The same applies to the technology used to produce Good 2 in Country 1, which means that Country 1 *effectively* has only two technologies. The same holds true for Country 2.

With each country possessing two technologies each, the total number of technologies is 4. As the facet of the 2-country, 2-goods economy is three-dimensional, it is spanned by 3 technologies. On the facet, one country utilizes 2 technologies, and since there is no inferior technology in this country, gains from trade do not arise. This explains why gains from trade do not occur for either country on the facet of the 2-country, 2-goods labor-input economy.

Figure 5 illustrates the efficient frontier of the 2-country, 2-goods labor-input economy. Country *A* produces both goods along the edge (a,b) and does not possess an inferior technology. Therefore, gains from trade do not occur in Country *A* along the edge (a,b). Along the edge (b,c), the same argument can be made for *B* country, and gains from trade do not occur for *B* country.

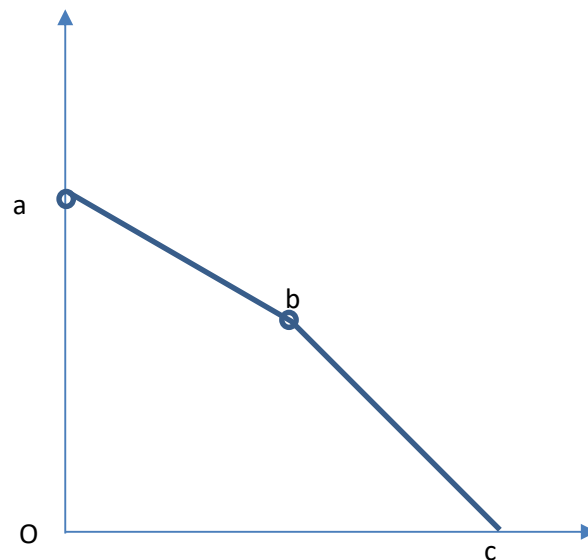


图 5 Efficient frontier of a 2-country, 2-goods labor-input economy

The fact that gains from trade do not occur for all countries on the facet is shown to depend on the extreme assumption of a 2-country, 2-goods labor-input economy.

Table 1 shows the technology matrix in a 2-country, 4-goods labor-input economy.

表 15 Numerical example of a 2-country 4-goods labor-input economy where trade benefits all countries on the facet.

Country of Affiliation for Technology	Country 1				Country 2			
Technology	$c_{(1)}$	$c_{(2)}$	$c_{(3)}$	$c_{(4)}$	$c_{(5)}$	$c_{(6)}$	$c_{(7)}$	$c_{(8)}$
Goods 1	1	0	0	0	1	0	0	0
Goods 2	0	1	0	0	0	1	0	0
Goods 3	0	0	1	0	0	0	1	0
Goods 4	0	0	0	1	0	0	0	1
Labor 1	1	1	1	1	0	0	0	0
Labor 2	0	0	0	0	1	1	1	1

The technologically feasible set for this economy consists of three facets:

$$\begin{aligned}
 F_I &:= (c_{(1)}, c_{(2)}, c_{(3)}, c_{(7)}, c_{(8)}), \\
 F_{II} &:= (c_{(2)}, c_{(3)}, c_{(4)}, c_{(5)}, c_{(6)}), \\
 F_{III} &:= (c_{(3)}, c_{(4)}, c_{(5)}, c_{(6)}, c_{(7)}).
 \end{aligned} \tag{12.9}$$

These facets are characterized by their respective frame matrices. To determine the rank of each frame matrix, one can use the function $rank(matrix)$ in Scilab. Table 16 presents the value associated with each facet. Since $q_I < 0$, F_I is not an eligible facet. Given $q_{II} > 0$, $q_{III} > 0$, F_{II} and F_{III} are eligible facets.

表 16 Value Associated with Each Facet

Value	p_1	p_2	p_3	p_4	w_1	w_2
q_I	-1	-0.923	-0.846	-0.846	-0.769	-0.846
q_{II}	1	1	0.917	0.833	0.833	1
q_{III}	1	1	1	0.909	0.909	1

Table 17 represents the benefits of each technology based on those values. For F_{II} , the inferior technology of Country 1 is $c_{(1)}$, while the inferior technologies of Country 2 are $c_{(7)}$ and $c_{(8)}$. In the case where both countries possess inferior technology, gains from trade occur for both nations.

In F_{III} , the inferior technology of Country 1 consists of $c_{(1)}$ and $c_{(2)}$, while Country 2's inferior technology is $c_{(8)}$. Even in this scenario, gains from trade materialize for both countries due to their possession of inferior technology.

表 17 Evaluation of technology benefits associated with each facet

Evaluation of technology benefits by each value	Technology	$c_{(1)}$	$c_{(2)}$	$c_{(3)}$	$c_{(4)}$	$c_{(5)}$	$c_{(6)}$
	q_I	0	0	0	-0.077	-0.153	-0.07
	q_{II}	-0.083	0	0	0	0	0
	q_{III}	-0.182	-0.091	0	0	0	0

Summarizing the above as a proposition.

【命題 62Number of Countries Experiencing Gains from Trade on the Facet of a Labor-Input Economy with 2 Countries and 15 Goods: For the occurrence of gains from trade on the facet of a labor-input economy with 2 countries and 15 goods, the following holds:

1. Country does not experience gains from trade if there are only 2 goods and under the assumption of a labor-input economy.
2. A numerical example can be constructed where gains from trade occur for both countries on two eligible facets in a labor-input economy with 2 countries and 4 goods.

13 Technological Change and Its Impact

13.1 Impact on Real Wage

In this section, we analyze the impact of technological change on real wages.

【定義 54Expansion of Technology: Technology set C_{II} is said to be an expansion of technology set C_I when technology set C_{II} contains and is not equal to technology set C_I , i.e., $C_{II} \supsetneq C_I$.

【仮定 3Content of Technological Change: Technological change satisfies the following two conditions:

1. The technology after the change is an enlargement of the technology before the change.
2. Before and after the technological change, wages remain constant and belong to the intersection of two positive wage ranges (see Definition 37).

Before the technological change, the technology matrix is denoted by the subscript O , and after the change, it is denoted by the subscript N (old and new, respectively). The goods prices are also distinguished before and after the technological change by p_O and p_N , respectively. Both are positive.

The profit from adopting the pre-technological change technology evaluated at p_O is zero:

$$p_O' (B_O - A_O) = w' L_O. \quad (13.1)$$

The image of (13.1) under the technology-goods mapping ϕ_O before the technological change is as follows:

$$p_O'(\mathbf{B}_O - \phi_O(\mathbf{A}_O)) = w'\phi_O(\mathbf{L}_O). \quad (13.2)$$

Some of the technologies in \mathbf{C}_O become unprofitable and cease operation after losing cost competitiveness due to technological changes.

【定義 55 Obsolete technology: A technology that was operational before a technological change but becomes inoperative after the change is referred to as *obsolete technology*.

The set of numbers for obsolete technologies is denoted by $T_O^{(Ob)}$ (obsolescence). The set of numbers for non-obsolete technologies in T_O is represented by $T_O^{(S)}$ (survival). Naturally, $T_O = T_O^{(Ob)} \cup T_O^{(S)}$, $T_O^{(Ob)} \cap T_O^{(S)} = \emptyset$. Evaluating the technologies adopted before technological changes in p_N , the profit for obsolete technology is negative, while the profit for non-obsolete technology is zero.

$$\text{For } \tau \in T_O^{(Ob)}, p_N' (b_{O(\tau)} - a_{O(\tau)}) < w'l_{O(\tau)}. \quad (13.3)$$

$$\text{For } \tau \in T_O^{(S)}, p_N' (b_{O(\tau)} - a_{O(\tau)}) = w'l_{O(\tau)}. \quad (13.4)$$

【定義 56 Product of obsolete technology or obsolete goods: Goods produced by obsolete technology before technological change are referred to as the product of obsolete technology or simply as obsolete goods.

The set of numbers for obsolete goods is represented by $N^{(Ob)}$, and the rest are denoted by $N^{(S)}$. Naturally, $N = N^{(Ob)} \cup N^{(S)}$. Without loss of generality, it can be assumed that $N^{(Ob)} = \{1, \dots, N_1\}$, $N^{(S)} = \{N_1 + 1, \dots, N\}$. Taking the image of equation (13.3) defined by ϕ_O .

$$\text{For } n \in N^{(Ob)}, p_N'(I - \phi_O(\mathbf{A}_O))_{(n)} < w'\phi_O(\mathbf{L}_O)_{(n)}. \quad (13.5)$$

The profit from non-obsolete technology that produces obsolete goods is zero, but it is weighted with the negative profit from obsolete technology producing the same goods, resulting in the average production profit of obsolete goods being negative. On the one hand, for $n \in N^{(S)}$, the price of the new good, $p_N'(I - \phi_O(\mathbf{A}_O))_n$, is equal to the wage multiplied by the production function of old goods, $w'\phi_O(\mathbf{L}_O)_n$.

From equations (13.2), (13.5), and (), it follows that for $n \in N^{(Ob)}$, the expression $(p_N' - p_O')(I - \phi_O(\mathbf{A}_O))_n < 0$.

For $n \in N^{(S)}$, when $(p_N' - p_O')(I - \phi_O(\mathbf{A}_O))_n = 0$, it can be inferred from equation () that the price of obsolete goods decreases. Moreover, even if the goods are not obsolete, if they are used as intermediate goods for obsolete goods, then the price decreases. This can be demonstrated as follows. The price change of goods n can be expressed as follows in the Japanese economic paper translated into English:

$$\begin{aligned}
(p'_N - p'_O)_n &= \left((p'_N - p'_O) (I - \phi_O(\mathbf{A}_O)) (I - \phi_O(\mathbf{A}_O))^{-1} \right)_n \\
&= (p'_N - p'_O) (I - \phi_O(\mathbf{A}_O)) (I - \phi_O(\mathbf{A}_O))^{-1}_{(n)} \\
&= \sum_{i \in N^{(Ob)}} ((p'_N - p'_O) (I - \phi_O(\mathbf{A}_O)))_{[i]} (I - \phi_O(\mathbf{A}_O))^{-1}_{i(n)} \\
&\quad + \sum_{i \in N^{(S)}} ((p'_N - p'_O) (I - \phi_O(\mathbf{A}_O)))_{[i]} (I - \phi_O(\mathbf{A}_O))^{-1}_{i(n)} \\
&= \sum_{i \in N^{(Ob)}} (p'_N - p'_O) (I - \phi_O(\mathbf{A}_O))_{(i)} (I - \phi_C(\mathbf{A}_C))^{-1}_{i(n)} \\
&\quad + \sum_{i \in N^{(S)}} (p'_N - p'_O) (I - \phi_O(\mathbf{A}_O))_{(i)} (I - \phi_C(\mathbf{A}_C))^{-1}_{i(n)} \\
&= \sum_{i \in N^{(Ob)}} (p'_N - p'_O) (I - \phi_O(\mathbf{A}_O))_{(i)} (I - \phi_C(\mathbf{A}_C))^{-1}_{i(n)}. \tag{13.6}
\end{aligned}$$

The first and fourth equalities utilize the fact that the product of the $\mathbf{X}\mathbf{Y}$ matrices' n columns is equal to the product of the \mathbf{X} and \mathbf{Y} n columns. The final equality utilized the fact that the second term of the fifth side is 0, as implied by Equation (). If goods n inputs obsolete goods as intermediate goods, then the term for those intermediate goods will be negative on the final side. This is because the product of the first and second factors is negative as per Equation (), and the third factor is positive because it is inputted as intermediate goods directly or indirectly. Therefore, the overall $(p'_O - p'_C)_n$ will also be negative.

【定義 57】Generalized Obsolete Goods: Goods that are obsolete in themselves and those produced by directly or indirectly using obsolete goods are referred to as generalized obsolete goods (see Fig. 6^{*14}).

^{*14} The interpretation of Fig. 6 is similar to that of Fig. 4 with necessary modifications, hence it is not reiterated here.

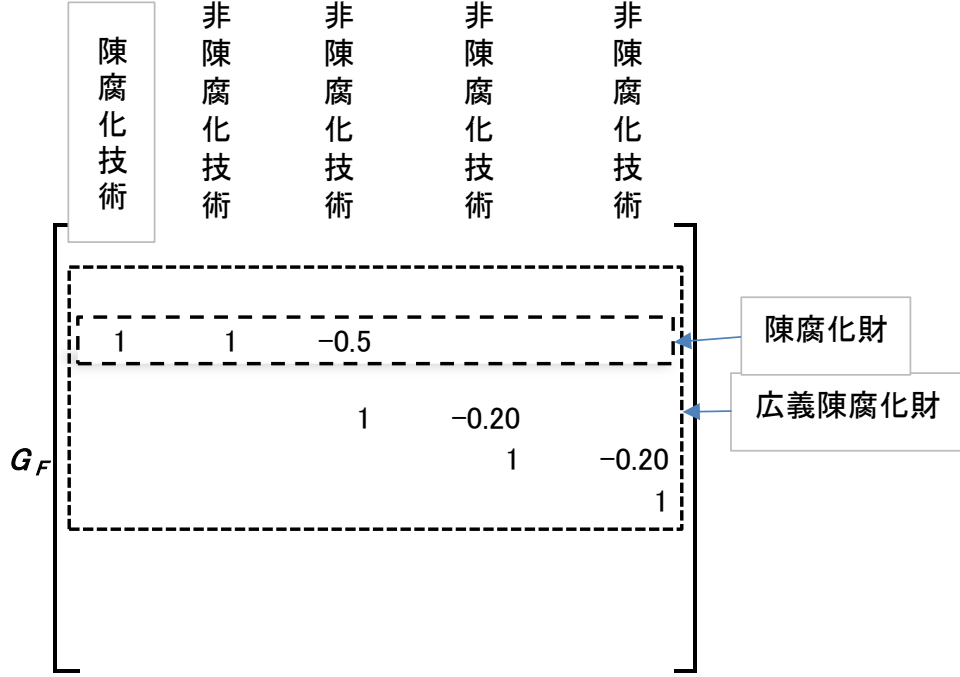


図 6 Obsolete Goods and Generalized Obsolete Goods

Therefore, we obtain 【Proposition 63】 .

【命題 63】 Changes in Real Wages:

1. The real wage measured by the price of generalized obsolete goods rises.
2. The real wage measured at prices other than generalized obsolete goods is equal.
3. There is no decrease in real wage regardless of the goods price measured after technological change.

13.2 Marginal Country

We examine the change in real wage of a country where there are no goods that can be produced at a constant nominal wage following a change in technology in other countries. Such a country is referred to as a marginal country^{*15}.

【定義 58 Marginal Country for Technological Change: A country where there are no goods that can be produced at a constant nominal wage following a change in technology in other countries is termed as the marginal country for technological change. It is assumed that the technology of the marginal country remains unchanged.

Let us consider a scenario where Country 1 is the marginal country and it produces goods 1

^{*15} An example of a marginal country in a 2-country 2-goods labor-input economy was cited in a presentation on July 2, 2023, by Professor Toshihiro Oka during the International Trade Theory Seminar on "The Meaning of Unemployment under Path Dependency."

using technology 1 before the technological change. We analyze how the real wage of Country 1 changes if the nominal wage w_1 needs to adjust to allow Country 1 to continue producing goods 1 post-technological change. The prices before and after the technological change are differentiated by subscripts O (old) and N (new). Under technology 1 in Country 1, where the technology $[p_N w_1]$ does not yield positive profits, denoted by subscript N ^{*16}. Country 1 can continue the production of goods 1 at the w_1 level denoted by \hat{w}_1 .

The price of goods 1 in Country 1 is represented as:

$$p_{1N} = p_N' a_{(1)} + \hat{w}_1 l_{1(1)}. \quad (13.7)$$

Since Country 1's Technology 1 lacks competitiveness at the old wage level, it does not affect the determination of the new price p_N . The condition for determining \hat{w}_1 in (13.7) is influenced by p_N .

The following equation expresses the relation:

$$\frac{\hat{w}_1}{p_{1N}} = \frac{1}{l_{1(1)}} \left(1 - \frac{p_N' a_{(1)}}{p_{1N}} \right). \quad (13.8)$$

The expression in the parentheses on the right-hand side represents the value-added ratio of Technology 1 evaluated at the new price (the ratio between the value after deducting the cost of intermediate goods from the goods price and the goods price itself), denoted as v_{1N} . A similar equation holds when transforming the price-cost equality equation of Technology 1 using the old wage level and old prices. The equality in both (13.8) and (13.9) holds because wages are changing.

Subtracting (13.8) from (13.9) yields the change in the real wage.

$$\frac{\hat{w}_1}{p_{1N}} - \frac{w_1}{p_{1O}} = \frac{1}{l_{1(1)}} (v_{1N} - v_{1O}). \quad (13.9)$$

If the price changes induced by technological advancements worldwide increase the value-added ratio of a marginal country's product, the real wage in that country will rise. Conversely, if the value-added ratio decreases, the real wage in the marginal country will fall.

From the price-cost equality equation of technology 1 evaluated at the old value ($p_O w_1$) and the new value ($p_N \hat{w}_1$), we obtain the following two equations^{*17}. These equations measure the real wage using a weighted average of the basket of prices with input coefficient $a_{(1)}$. The translation of the provided text is as follows:

$$\frac{\hat{w}_1}{p_N' a_{(\tau)}} = \frac{1}{l_{1(1)}} \left(\frac{p_{1N}}{p_N' a_{(\tau)}} - 1 \right) = \frac{1}{l_{1(1)}} \left(\frac{1}{1 - v_{1N}} - 1 \right), \quad (13.10)$$

$$\frac{w_1}{p_O' a_{(\tau)}} = \frac{1}{l_{1(1)}} \left(\frac{1}{1 - v_{1O}} - 1 \right). \quad (13.11)$$

^{*16} The subscript N in p_N represents 'New' rather than an element number.

^{*17} The final equality in the first equation of (13.11) follows from $v_{1N} = 1 - p_N' a_{(1)}/p_{1N} \Rightarrow p_N' a_{(1)}/p_{1N} = 1 - v_{1N} \Rightarrow p_{1N}/p_N' a_{(1)} = (1 - v_{1N})^{-1}$.

Taking the difference of the two equations, we have

$$\frac{\hat{w}_1}{p'_N a_{(\tau)}} - \frac{w_1}{p'_O a_{(\tau)}} = \frac{1}{l_{1(1)}} \left(\frac{1}{1 - v_{1N}} - \frac{1}{1 - v_{1O}} \right). \quad (13.12)$$

It is found that the real wage changes in the same direction as the value-added ratio when measured by the prices of a basket weighted by input coefficients. This observation can be summarized as a proposition.

【命題 64 Changes in Real Wages for Marginal Country: When global technology changes, and the prices of goods and nominal wages in the marginal country change, the real wage in the marginal country changes in the same direction as the value-added ratio. Whether measuring real wages based on the prices of goods produced by the marginal country or the basket of intermediate goods, the conclusion remains the same.

It is easier to understand when represented in a diagram.

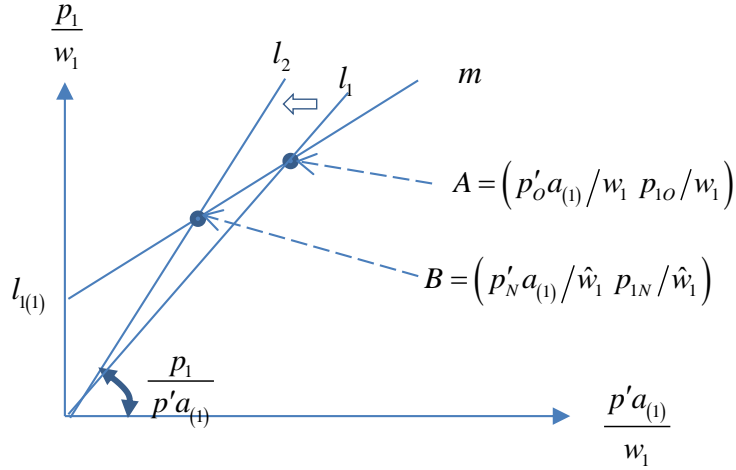


図 7 Changes in Real Wages for Marginal Country

The horizontal axis in Figure 7 represents the value of the input goods basket relative to w_1 . The vertical axis represents the price of goods 1 relative to w_1 . The line represented by m depicts the price-cost equalization condition. The slope of the line m is 1, and the value where it intersects the vertical axis corresponds to the labor input quantity $l_{1(1)}$. The equation of line m is given by (13.13).

$$\frac{P_1}{W_1} = \frac{P' A_{(1)}}{W_1} + L_{1(1)}. \quad (13.13)$$

The two states before and after technological change, $A = (p'_O a_{(1)}/w_1, p_{1O}/w_1)$ and $B = (p'_N a_{(1)}/\hat{w}_1, p_{1N}/\hat{w}_1)$, both lie on the line segment m as they satisfy the condition of price-cost equalization.

Points on the straight line passing through the origin with a slope of $p_1/p'a_{(1)}$ represent $(p'a_{(1)}/w_1, p/w_1)$. The intersection of this line with line m determines the real wage. Given that the value-added ratio is $(1 - p'a_{(1)}/p_1) = (1 - 1/\text{傾き})$, an increase in the value-added

ratio leads to an increase in the slope. Figure 7 illustrates that shifting the line from l_1 to l_2 brings the intersection closer to the origin, resulting in an increase in the real wage (reciprocal of both axes).

A decrease in the value-added ratio of goods in the marginal country occurs when the prices of goods produced by that country decrease compared to the weighted average prices of other goods (weighted by the marginal country's input coefficients of technology). Considering that as the country loses the ability to produce certain goods, the prices of those goods decrease, it is natural to assume that the value-added ratio also decreases. Therefore, it is reasonable to expect a decrease in the real wage.

However, cases like that of the marginal country (losing the ability to produce all goods due to technological change in other countries) seem to be exceptional. Regarding the primary effects of technological change, it is considered plausible to apply the conclusions for the presence of generalized obsolete goods from Proposition 63 under the assumption of constant wages.

13.3 Impact on Employment

【命題 65 Technological Change and Full Employment: For a non-marginal country, there exists an operating scale that can achieve full employment while maintaining the wage level from before the technological change.

Proof: Let the same wage level before and after technological change be denoted as w . Assume w belongs to the intersection of two positive wage ranges before and after the technological change. Denote the maximal technology face corresponding to w after technological change as F . Let p represent the price associated with w on F . The maximization problem

$$\max_x q'y \quad \text{s.t.} \quad y = C_F x, \quad l_F x \leq \hat{l}, \quad x \geq 0 \quad (13.14)$$

has a solution. This solution utilizes all labor endowments and constitutes a state of full employment. (End of proof)

14 How to Close or Not Close the Model

14.1 Overview of the Model

Figure 8 illustrates the overall structure of the model presented in this paper. The given conditions consist of the technology matrix C , nominal wage w , the labor endowments of each country \hat{l} , independent demand I , and the world demand function D as functions of I and p .

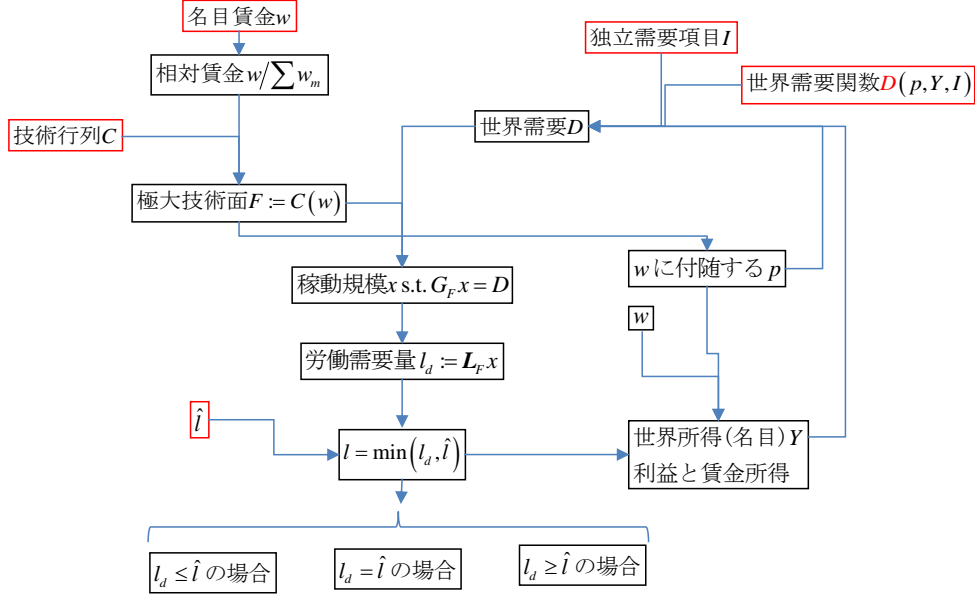


図 8 Overview of the Model

Given the wage w , the relative wage $w / \sum w_m$ is determined. This relative wage, combined with the given technology matrix C , determines the maximal technology face $C(w)$ and the associated price p .

The world demand, employment, and world nominal income are determined simultaneously. The mechanism is as follows: world demand determines the quantity of labor demand, which in turn determines the actual level of employment by taking the smaller value between labor demand and labor endowments. The actual level of employment and wages determine the world nominal income. The world nominal income, associated price, and independent demand I determine the world demand.

The closure of the model is determined by considering what happens when labor demand quantity differs from labor endowments.

14.2 Closing of Equilibrium Theory

The closing problem of models based on equilibrium theory lies in the inability to provide a convincing explanation of how the economy returns to an equilibrium solution when it deviates from equilibrium. Let's consider a scenario where unemployment leads to a decrease in wages, causing prices to decrease according to the markup principle. In this case, with a decrease in nominal revenue, companies may face difficulties in repaying long-term debts. Consequently, bankruptcies may occur, leading to a decrease in investment and a negative spiral in real demand.

Furthermore, in real labor markets, the movement of wages does not necessarily follow a pattern of decreasing wages in response to unemployment.

The closing approach of equilibrium theory models significantly deviates from the reality of the economy.

14.3 An "Adaptive" Perspective

Let's observe the real economy. Companies meticulously study latent demand when introducing new commodities. This indicates that the technology matrix and world demand function are not unrelated but closely intertwined in their development.

When demand is expected to exceed supply capacity, companies order facilities to expand capacity before actual excess demand occurs. If labor supply constraints are the primary cause, labor-saving equipment is constructed. In the case of Figure 8, there exists an arrow pointing towards the independent demand element I.

In practice, when demand reaches its supply limit, the situation becomes as follows.

1. When the supply reaches its limit at the standard markup rate and excess demand occurs, some of the demand goes into a waiting state. Buyers who cannot wait propose a price increase and negotiate with suppliers for preferential allocation. In response, some suppliers temporarily sell a portion of output quantity at a higher price (higher markup rate).
2. This chain reaction propagates through the entire economy via the input-output relationships of goods.
3. The degree of price pass-through at each stage critically depends on the proportion of buyers who cannot wait at each stage.
4. Factors determining the proportion of buyers who cannot wait at each stage include the pathway through which the proportion of buyers who cannot wait at the final demand stage ripples upstream. Various other factors could also be considered.

Conversely, when demand falls below the supply capacity, the response is not price reduction but production cutbacks. The speed at which employment adjusts when the actual employment level falls below the labor demand quantity depends on the institutional characteristics of each country's labor markets. Japan exhibits a very slow adjustment speed, while the United States shows a faster adjustment speed. While wage reductions may occur when unemployment arises, they may not decrease at the pace required by equilibrium theory.

At present, we cannot provide a model that is consistent with such perspectives, but it is considered important to continue research in this direction rather than aiming to fully close the model.

15 Illustration using a 2-country, 2-goods labor-input economy

15.1 Technology Matrix

We designate the two countries as Country A and Country B. There are two types of goods, and there are no intermediate goods inputs (assuming a labor-input economy). Each country possesses one technology for producing each good (goods 1 and goods 2). The technology matrix is normalized by labor input quantity. Contrary to previous notation, we represent the technology vectors for Country A and Country B as vectors a and b , respectively. To distinguish between two technologies in the same country, assign a subscript to the goods number produced by the technology. For instance, $a_{(1)}$ represents the technology of country A producing goods 1.

表 18 Technology Matrix of 2-country, 2-goods Labor Input Type

	Country A		Country B	
Technology	$a_{(1)}$	$a_{(2)}$	$b_{(1)}$	$b_{(2)}$
Goods 1	a_1	0	b_1	0
Goods 2	0	a_2	0	b_2
Labor 1	-1	-1	0	0
Labor 2	0	0	-1	-1

Country A possesses a comparative advantage in goods 1: This implies the following. To increase goods 1 by a_1 units, Country A must decrease goods 2 by a_2 units. The opportunity cost of goods 1 in terms of goods 2 is a_2/a_1 . Similarly, the opportunity cost of goods 1 for Country B is b_2/b_1 . The opportunity cost of goods 1 for Country A is lower than that of Country B, indicating that Country A holds a comparative advantage in goods 1.

$$\frac{a_2}{a_1} < \frac{b_2}{b_1}. \quad (15.1)$$

The same principle can be expressed as a comparison of productivity for each good.

$$\frac{b_1}{a_1} < \frac{b_2}{a_2}. \quad (15.2)$$

Country A exhibits higher relative productivity in producing goods 1 compared to goods 2.

Assuming that Country B has absolute advantage in both goods. This assumption serves a dual purpose. Firstly, it enhances the clarity of the graph. Secondly, it highlights that this assumption plays no role in the analysis; emphasizing that relative productivity, not absolute

productivity, is the crucial factor.

$$\begin{aligned} b_1 &> a_1, \\ b_2 &> a_2. \end{aligned} \tag{15.3}$$

15.2 Technology Face and Value Face

Given that $M + N = 4$, we analyze pair matches of a 3D technology face (facet) with a 1D value face, and pair matches of a 2D technology face with a 2D value face.

15.2.1 3D Technology Face and 1D Value Face

A 3D technology face is a combination of 3 out of 4 columns from a 4×4 technology matrix

$$\begin{bmatrix} a_{(1)} & a_{(2)} & b_{(1)} & b_{(2)} \end{bmatrix}. \tag{15.4}$$

These faces are formed by this submatrix, and include four categories of pair matches. Translated economic paper from Japanese written in text:

In accordance with Equation , which is comprised of four terms as indicated: $(a_{(2)} \ b_{(1)} \ b_{(2)})$, $(a_{(1)} \ b_{(1)} \ b_{(2)})$, $(a_{(1)} \ a_{(2)} \ b_{(2)})$, and $(a_{(1)} \ a_{(2)} \ b_{(1)})$, which respectively represent polyhedral cones, they are sequentially assigned numbers as "3-dimensional technology face1," "3-dimensional technology face2."

For the "3-dimensional technology face1," the 1-dimensional value face (also referred to as value half-line) denoted as (q) is a solution satisfying the following system of simultaneous equations and linear inequalities.

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4, \end{bmatrix} \begin{bmatrix} 0 & b_1 & 0 \\ a_2 & 0 & b_2 \\ -1 & 0 & 0, \\ 0 & -1 & -1 \end{bmatrix} = 0 \tag{15.5}$$

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} a_1 & 0 & -1 & 0 \end{bmatrix}' \leq 0. \tag{15.6}$$

Expanding this:

$$a_2 q_2 - q_3 = 0, \tag{15.7}$$

$$b_1 q_1 - q_4 = 0, \tag{15.8}$$

$$b_2 q_2 - q_4 = 0, \tag{15.9}$$

$$a_1 q_1 - q_3 \leq 0. \tag{15.10}$$

Solving this system of equations for q_4 , we get the following results: From the second equation, $q_1 = q_4/b_1$. From the third equation, $q_2 = q_4/b_2$. Substituting these into the first equation yields $q_3 = a_2 q_2 = (a_2/b_2) q_4$. The final inequality is given by Equation 2247, but substituting the solution to Equation 2248 yields

$$\left(\frac{A_2}{B_2} - \frac{A_1}{B_1} \right) Q_4 \geq 0. \tag{15.11}$$

Dividing by the comparative advantage condition (15.1), we have

$$Q_4 \leq 0. \quad (15.12)$$

We use $q_4 = -1$ as representative values:

$$Q_1 = -\frac{1}{B_1}, \quad Q_2 = -\frac{1}{B_2}, \quad Q_3 = -\frac{A_2}{B_2}, \quad Q_4 = -1. \quad (15.13)$$

Similar methods can be used to solve equations for other technology faces leading to results summarized in Table 19^{*18}. In Table 19, 7-th column the "eligibility" indicates a fully eligible technology face for \bigcirc , while \times denotes an ineligible technology face. The same applies to other tables.

表 19 3D Technology Face and 1D Value Face

Number of 3D Technology Face	1D Value Face	Elements				Eligibility
		q_1	q_2	q_3	q_4	
1	q_I	$-\frac{1}{b_1}$	$-\frac{1}{b_2}$	$-\frac{a_2}{b_2}$	-1	\times
2	q_{II}	$\frac{1}{b_1}$	$\frac{1}{b_2}$	$\frac{a_1}{b_1}$	1	\bigcirc
3	q_{III}	$\frac{1}{a_1}$	$\frac{1}{a_2}$	1	$\frac{b_2}{a_2}$	\bigcirc
4	q_{IV}	$-\frac{1}{a_1}$	$-\frac{1}{a_2}$	-1	$-\frac{b_1}{a_1}$	\times

When the absolute value of q_4 is set to 1, it becomes easier to plot the graph. For visual representation, Table 20 should be utilized.

表 20 Graph for 3D technology face and 1D value face

Number of 3D technology face	1D value face	Elements				Eligibility
		q_1	q_2	q_3	q_4	
1	q_I	$-\frac{1}{b_1}$	$-\frac{1}{b_2}$	$-\frac{a_2}{b_2}$	-1	\times
2	q_{II}	$\frac{1}{b_1}$	$\frac{1}{b_2}$	$\frac{a_1}{b_1}$	1	\bigcirc
3	q_{III}	$\frac{a_2}{a_1} \frac{1}{b_2}$	$\frac{1}{b_2}$	$\frac{a_2}{b_2}$	1	\bigcirc
4	q_{IV}	$-\frac{1}{b_1}$	$-\frac{a_1}{a_2} \frac{1}{b_1}$	$-\frac{a_1}{b_1}$	-1	\times

15.2.2 2D Technology Face and 2D Value Face

The intersection of two 3D technology faces with two common technologies results in a 2D technology face. The intersection of technology face1

$$(a_{(2)} \ b_{(1)} \ b_{(2)})$$

^{*18} see Appendix1 Derivation of 3-Dimensional Technology Face and 1-Dimensional Value Face' for details

and technology face²

$$(a_{(1)} \ b_{(1)} \ b_{(2)})$$

is

$$(b_{(1)} \ b_{(2)})$$

and its 2-dimensional value face is $(q_I \ q_{II})$. There are ${}_4C_2 = 4!/(2!2!) = (4 \cdot 3 \cdot 2 \cdot 1)/(2 \cdot 2) = 6$ pairs of matches. However, $(q_I) = (-q_{III})$ および $(q_{II}) = (-q_{IV})$, the dimensions of $(q_I \ q_{III})$ and $(q_{II} \ q_{IV})$ are not 2-dimensional, but 1-dimensional, and they are excluded. All 2-dimensional technology faces are as shown in Table 21. The \triangle in the "eligibility" column of Table 21 refers to incompletely eligible technology faces, which are incomplete. This holds true for other tables as well.

表 21 Two-dimensional technology face and two-dimensional value face

Pairing of three-dimensional technology face	, two-dimensional technology face	, two-dimensional value face
1, 2	, $(b_{(1)} \ b_{(2)})$, $(q_I \ q_{II})$
1, 4	, $(a_{(2)} \ b_{(1)})$, $(q_I \ q_{IV})$
2, 3	, $(a_{(1)} \ b_{(2)})$, $(q_{II} \ q_{III})$
3, 4	, $(a_{(1)} \ a_{(2)})$, $(q_{III} \ q_{IV})$

There are more eligible two-dimensional technology faces than usually mentioned. Adding the condition that all countries produce, faces $(b_{(1)} \ b_{(2)})$ and $(a_{(1)} \ a_{(2)})$ are eliminated, leaving only the fully eligible technology face, $(a_{(1)} \ b_{(2)})$.

15.2.3 1-Dimensional Technology Face and 3-Dimensional Value Face

The 1-dimensional technology face consists of a single technology. There are four of them: $(a_{(1)})$, $(a_{(2)})$, $(b_{(1)})$, and $(b_{(2)})$. The 3-dimensional value face is a polyhedral cone spanned by the value faces of 3-dimensional technology faces, including the 1-dimensional technology face. For example, since $(a_{(1)})$ contains the following three 3-dimensional technology faces

$$(a_{(1)} \ b_{(1)} \ b_{(2)}), (a_{(1)} \ a_{(2)} \ b_{(2)}), (a_{(1)} \ a_{(2)} \ b_{(1)})$$

, the value face of $(a_{(1)})$ is $(q_{II} \ q_{III} \ q_{IV})$. Table 22 presents the 1-dimensional technology face and 3-dimensional value face.

表 22 One-dimensional Technology Face and Three-dimensional Value Face

One-dimensional Technology Face	Three-dimensional Technology Face Number including One-dimensional
$(a_{(1)})$	2, 3, 4
$(a_{(2)})$	1, 3, 4
$(b_{(1)})$	1, 2, 4
$(b_{(2)})$	1, 2, 3

15.3 Eligible Technology Face

The eligible technology face consists of two three-dimensional faces, five two-dimensional faces, and four one-dimensional faces. Among these, those satisfying the two conditions of (i) each country producing at least one good and (ii) all goods being produced, have two three-dimensional faces (with two being fully eligible technology faces), one two-dimensional face, and zero one-dimensional faces. When adding these two conditions, all eligible technology faces become fully eligible technology faces.

15.4 Wage Face, Price Face, Technology Face

We only consider the fully eligible technology face and the associated value face. The value face, price face, wage face, and fully eligible technology face are as shown in Table 23. The interpretation of Table 23 is as follows: When the elements within parentheses are separated by commas as in (x_1, x_2) , they represent the coordinates of points in E^2 , distinguishing them from the polyhedral cone notation spanned by the elements. Additionally, the square brackets denote a matrix with column vectors represented as columns. The subscript L under the square brackets denotes the labor components of the technology matrix, representing rows 3 and 4, while G denotes the commodity components of the technology matrix, representing rows 1 and 2.

表 23 Wage Face, Price Face, Technology Face

Dimension	Value Face	Wage Face	Price Face	Technology Face
1st Dimension	q_{II}	$w_{II} := ((a_1/b_1, 1))$	$p_{II} := w'_{II} [a_{(1)} b_{(1)} b_{(2)}]_L \times [a_{(1)} b_{(1)} b_{(2)}]_G^{-1}$	$(a_{(1)} b_{(1)} b_{(2)})$
	q_{III}	$w_{III} := ((1, b_2/a_2)) = ((a_2/b_2, 1))$	$p_{III} := w'_{III} [a_{(1)} a_{(2)} b_{(2)}]_L \times [a_{(1)} a_{(2)} b_{(2)}]_G^{-1}$	$(a_{(1)} a_{(2)} b_{(2)})$
2nd Dimension	$(q_{II} q_{III})$	$(w_{II} w_{III})$	$(p_{II} p_{III})$	$(a_{(1)} b_{(2)})$

Figure illustrates the wage face and price face of the 1st dimension value face. The figure illustrates the 1-dimensional wage face and price face.

Figure depicts the commodity component and labor component of the 3-dimensional technology face.

Further explanation may be necessary to understand the viewpoint of the 3-dimensional technology face displayed in figure . \odot shows the vectors of the commodity component of three technologies forming the 3-dimensional technology face2. These vectors are defined as $(a_1, 0)'$, $(b_1, 0)'$, and $(0, b_2)'$. $(a_{(1)} \ b_{(1)} \ b_{(2)})_G$ is the polyhedral cone spanned by these three vectors.

Similarly, \square demonstrates the vectors of the commodity component of three technologies forming the 3-dimensional technology face3. The polyhedral cone spanned by these three vectors is represented as $(a_{(1)} \ a_{(2)} \ b_{(2)})_G$.

Although the technology faces in all dimensions are different, when we observe the projection onto the goods space, they all spread over the entire non-negative orthant of the goods space, making it difficult to distinguish each technology face. The same situation applies to the labor component, where everything spreads over the entire non-positive quadrant of the labor space, and the distinction between each technology face is not clear.

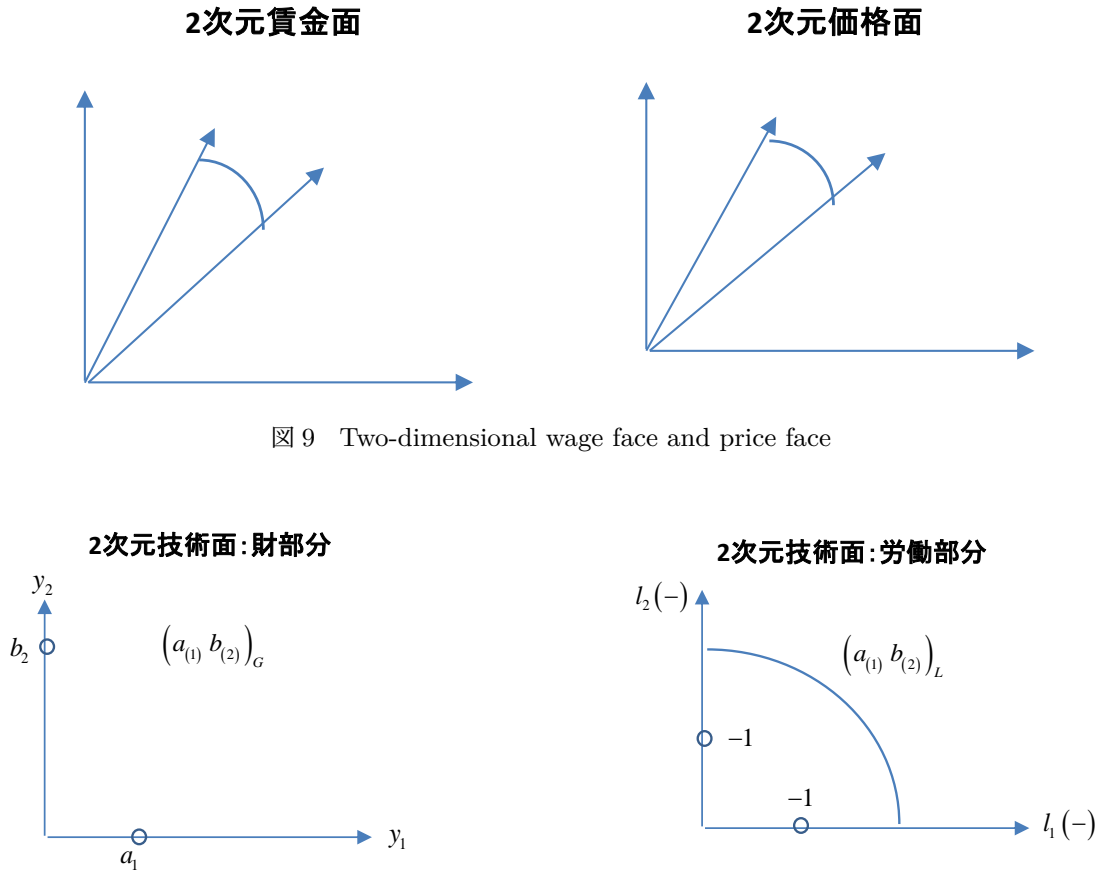


図 9 Two-dimensional wage face and price face

図 10 Commodity component and labor component of the two-dimensional technology face

To visually recognize the differences in technology faces by reducing one dimension and drawing

in a three-dimensional space, we fix the labor input of country B at a certain value l_2 and plot the graph in the coordinate system of $y_1 \cdot y_2 \cdot l_1$. For the derivation method, refer to "Section Appendix2, the technology face with fixed l_2 ." Figure 11 illustrates a two-dimensional technology face ($a_{(1)} b_{(2)}$). With the labor input fixed at a value l_2 , the output quantity of goods 2 remains constant at \bar{l}_2 . As l_1 changes, the y_1 varies along an upward-sloping straight line on the plane H .

図 11 A two-dimensional technology face with a fixed l_2

Moving on to Figure 12, it presents a three-dimensional technology face2. The right panel shows a projection onto the $y_1 \cdot y_2$ plane. In the case of $l_1 = 0$, the diagram consists of a downward-sloping straight line segment near the origin, and as l_1 increases, the slope remains constant while shifting upward in parallel.

The depiction on the right is a redrawn version in the $y_1 \cdot y_2 \cdot l_1$ coordinate system, shown on the left. When $l_1 = 0$, there exists a downward-sloping straight line segment in the lower left of the $y_1 \cdot y_2$ plane. As l_1 increases, the trajectory moves up and to the right in the $y_1 \cdot y_2 \cdot l_1$ coordinate system, creating a faint navy-blue sloping face.

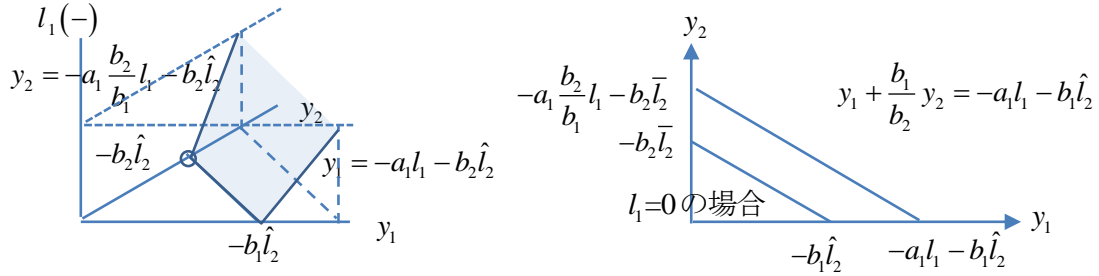


図 12 3-dimensional technology face2 with fixed l_2

Figure 13 illustrates the 3-dimensional technology face3. The right graph is a projection onto the $y_1 \cdot y_2$ plane. For $l_1 = 0$, the graph is a lower rightwards straight line interval close to the origin, and as l_1 increases, the slope remains constant while shifting parallel upwards. To highlight the differences with technology face2, the graph of technology face2 is shown with dashed lines.

The graph on the right is redrawn within the $y_1 \cdot y_2 \cdot l_1$ coordinate system on the left. For $l_1 = 0$, it is a lower rightwards straight line interval on the $y_1 \cdot y_2$ plane. As l_1 increases, the graph shifts to the upper right within the $y_1 \cdot y_2 \cdot l_1$ coordinate system, tracing a faint navy diagonal face.

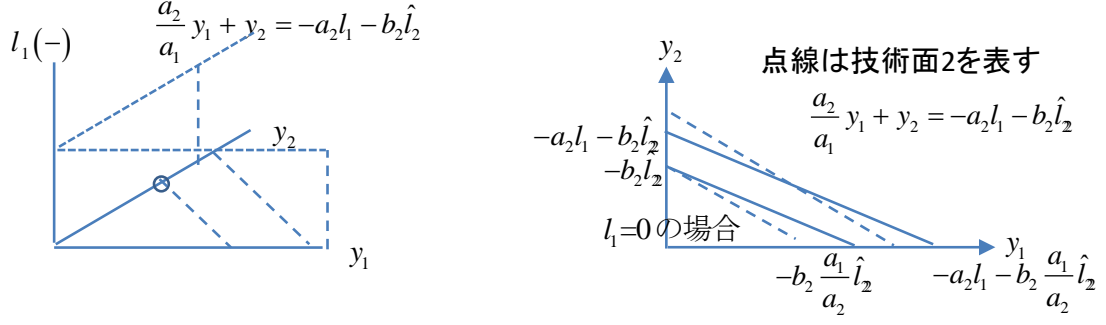


図 13 3-dimensional technology face3 with fixed l_2 : technology face2 shown with dashed lines

Figure 14 displays a graph overlay of 3-dimensional technology face2 and 3. By analyzing this figure, the clear distinction between two technology faces is evident.

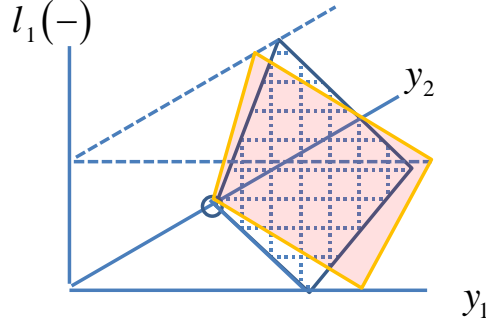


図 14 3-dimensional technology face 2 and 3 with fixed l_2

15.5 Minimal Wage Face and Maximal Technology Face

Eligible technology faces consist of five categories. Among 2-dimensional technology faces, there are three: $(b_{(1)} \ b_{(2)})$, $(a_{(1)} \ b_{(2)})$, and $(a_{(1)} \ a_{(2)})$ (see Table 21). Excluding the unique case where only one country produces $(b_{(1)} \ b_{(2)})$ and $(a_{(1)} \ a_{(2)})$, the remaining one is $(a_{(1)} \ b_{(2)})$. For 3-dimensional technology faces, there are two: the second $(b_{(1)} \ b_{(2)})$ and the third $(a_{(1)} \ a_{(2)} \ b_{(2)})$ (see Table 19).

Figure 15 illustrates four 1-dimensional wage faces: w_I , w_{II} , w_{III} , and w_{IV} . Table 24 demonstrates how minimal wage face and maximal technology face change based on the position of w .

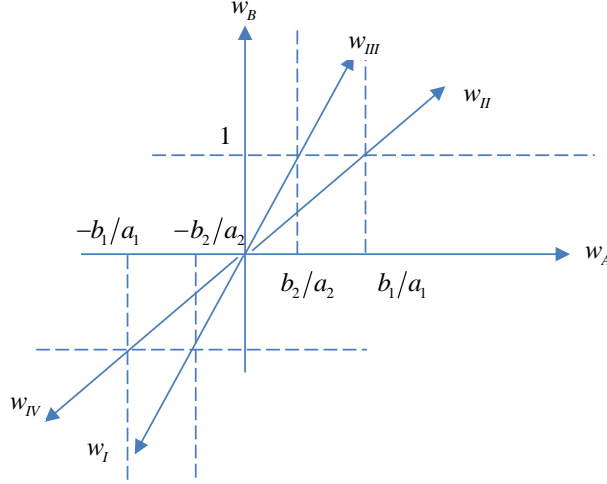


图 15 Four 1-dimensional wage faces

表 24 Minimal Wage Face and Maximal Technology Face

Position of $w > 0$ (counterclockwise)	Minimal wage face	Maximal technology face	Types of eligible tech
Inside ($w_I w_{II}$)	$(w_I w_{II})$	$(b_{(1)} b_{(2)})$	Incomplete
Above (w_{II})	(w_{II})	$(a_{(1)} b_{(1)} b_{(2)})$	Complete
Inside ($w_{II} w_{III}$)	$(w_{II} w_{III})$	$(a_{(1)} b_{(2)})$	Complete
Above (w_{III})	(w_{III})	$(a_{(1)} a_{(2)} b_{(2)})$	Complete
Inside ($w_{III} w_{IV}$)	$(w_{III} w_{IV})$	$(a_{(1)} a_{(2)})$	Incomplete

15.6 Reachable Region of Technology Face^{*19}

In order to determine the attainable commodity components by country within the technology face, the vector sum of these components represents the attainable region of the technology face. This explanation is limited to eligible technology faces.

15.6.1 Attainable Region of a 3-Dimensional Technology Face

For the country A in the 3-dimensional technology face 2, the attainable commodity component $(a_{(1)} b_{(1)} b_{(2)})$ (where the superscript denotes the dimension and the subscript denotes the technology number and country identifier) is given by:

$$P_{IIA}^{(3)} = \left\{ (a_1 x_1, 0)' \mid 0 \leq x_1 \leq \hat{l}_A \right\} \quad (15.14)$$

^{*19} In this section and the following "Face Movement" section, we do not distinguish between C and \hat{C} for simplicity in the figures.

The attainable region for country B in the 3-dimensional technology face 2, denoted as $P_{IIB}^{(3)}$, is defined as:

$$P_{IIB}^{(3)} = \left\{ (b_1 x_3, b_2 x_4)' \mid 0 \leq x_3 + x_4 \leq \hat{l}_B, (x_3, x_4) \geq 0 \right\} \quad (15.15)$$

The attainable region of the 3-dimensional technology face 2, denoted as $P_{II}^{(3)}$, can be expressed as the sum of the individual attainable regions for country A and country B:

$$P_{II}^{(3)} = P_{IIA}^{(3)} + P_{IIB}^{(3)} \quad (15.16)$$

Using Figure 16, we illustrate the process of constructing the attainable region of the 3-dimensional technology face2. The interval from 0 to $a_1 \bar{l}_A$ on the y_1 axis is denoted by $P_{IIA}^{(3)}$, forming a triangle $0 \cdot b_1 \hat{l}_B \cdot b_2 \hat{l}_B$. The vector sum of the two sets is $P_{II}^{(3)}$, constituting the attainable region of the 3-dimensional technology face2.

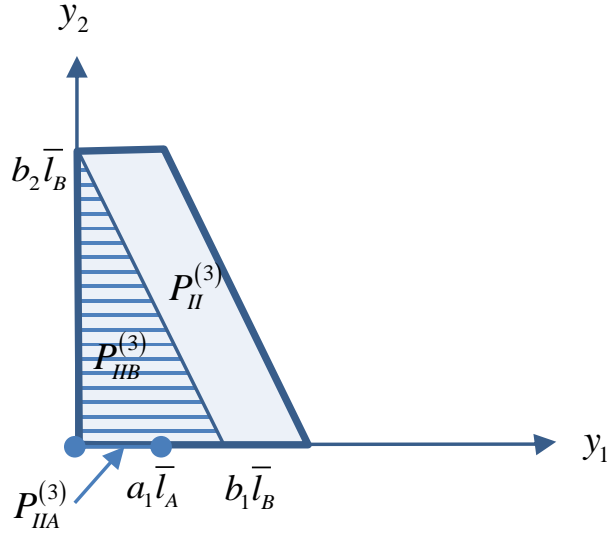


图 16 可达区域 of the commodity component of 3-dimensional technology face2

The attainable commodity component of country A in 3-dimensional technology face3 $(a_{(1)} \ a_{(2)} \ b_{(2)})$ is given by

$$P_{IIIA}^{(3)} = \left\{ (a_1 x_1, a_2 x_2)' \mid 0 \leq x_1 + x_2 \leq \hat{l}_A, (x_1, x_2) \geq 0 \right\}. \quad (15.17)$$

The attainable commodity component $P_{IIIB}^{(3)}$ of country B in technology face 3 is given by

$$P_{IIIB}^{(3)} = \left\{ (0, b_2 x_4)' \mid 0 \leq x_4 \leq \hat{l}_B \right\} \quad (15.18)$$

The attainable commodity component $P_{III}^{(3)}$ in technology face 3 is determined as

$$P_{III}^{(3)} = P_{IIIA}^{(3)} + P_{IIIB}^{(3)}. \quad (15.19)$$

Figure 17 illustrates, similar to Figure 16, the relationship between the attainable regions of each country and the overall attainable region of technology face 3.

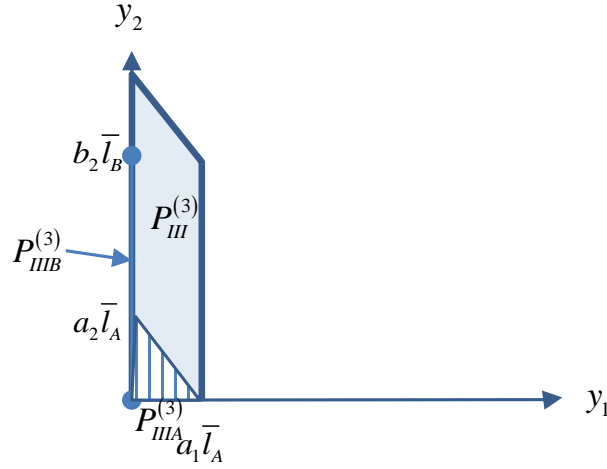


Figure 17 Attainable region of the commodity component of technology face 3 in 3 dimensions

15.6.2 Attainable Region of 2-Dimensional Technology Face

The attainable commodity components $P_A^{(2)}, P_B^{(2)}$ for countries A and B in technology face 3, a 2-dimensional representation, are as follows: The translation of the Japanese economic paper written in tex is as follows:

$$P_A^{(2)} = \left\{ (a_1 x_1, 0)' \mid 0 \leq x_1 \leq \hat{l}_A \right\}, \quad (15.20)$$

$$P_B^{(2)} = \left\{ (0, b_2 x_4)' \mid 0 \leq x_4 \leq \hat{l}_B \right\}. \quad (15.21)$$

The attainable set of the 2-dimensional technology face P is represented as:

$$P^{(2)} = P_A^{(2)} + P_B^{(2)} \quad (15.22)$$

Figure 18 illustrates the relationship between the attainable sets of each country and the overall attainable set of the technology face.

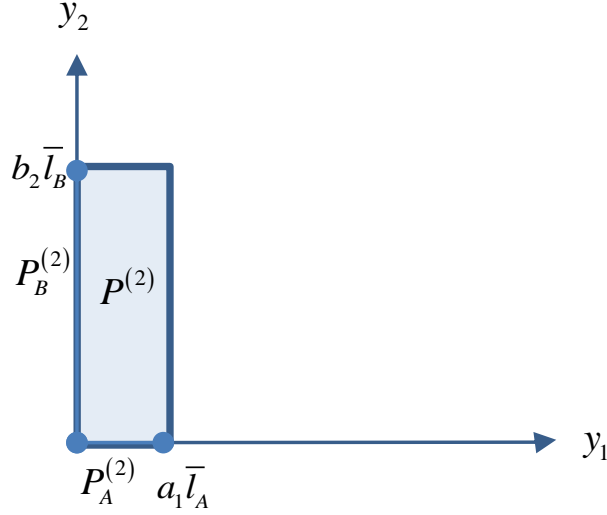


図 18 Attainable set of the commodity component of the 2-dimensional technology face

15.7 Face Movement Part 1

Using figures from 19 to , we explain the mechanism by which the economy moves to a different technology face when demand changes.

Figure 19 represents the initial state. Wages are determined by $w(0)$, and the minimum wage face $W(w(0))$ is a two-dimensional wage face (w_{II} w_{III}), while the maximal technology face is a two-dimensional technology face ($a_{(1)}$ $b_{(2)}$). It is assumed that the economy is located within the interior of the reachable region of the maximal technology face ($a_{(1)}$ $b_{(2)}$).

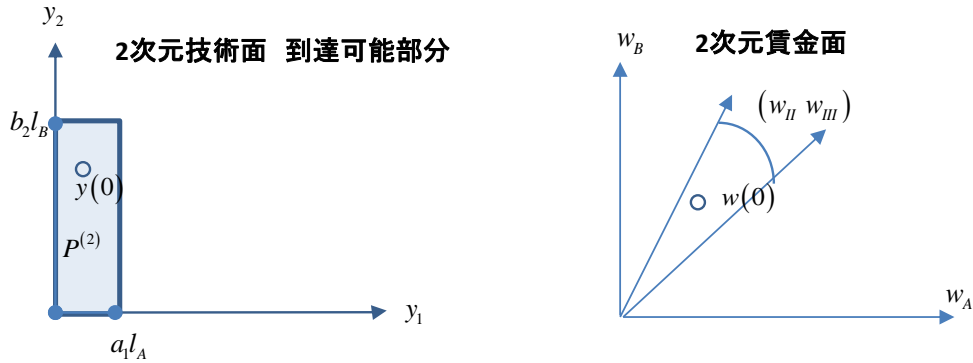


図 19 State inside the reachable region of the commodity component of the 2-dimensional technology face

As the demand for goods2 increases, the production of goods2 in country B increases, and the economy moves from $y(0)$ to the production upper limit of goods2, $y(1)$. During this transition, wages remain unchanged at $w(0)$ (see Figure 20).

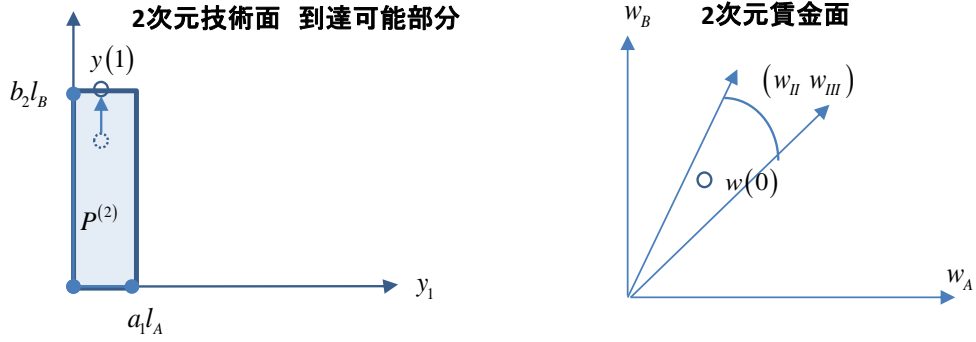


図 20 Movement to the boundary of the reachable region of the commodity component of the 2-dimensional technology face

As country B reaches full employment and competition for labor intensifies to meet the demand for goods2, wages in country B rise toward $w(1)$. During this period, $y(1)$ remains unchanged on the technology face, as goods2 cannot be increased (see Figure).

As the wage reaches $w(1)$, the minimal wage face transitions to (w_{III}) (shrinks), while the technology face expands to a 3-dimensional space, denoted as technology face3 at $(a_{(1)} a_{(2)} b_{(1)})$. Although the economy still resides at $y(1)$, the feasible region enlarges from $P^{(2)}$ to $P_{III}^{(3)}$ (refer to Figure). With the expansion of the maximal technology face to $P_{III}^{(3)}$, the economy can move in any direction from $y(1)$, while the wage remains at $w(1)$.

The summarized depiction of these processes is presented in Table 25.

表 25 Transition from 2D technology face to 3D technology face.

Period	State	Wage	Output quantity	Minimal wage face	Maximal technology face
0	Initial state	$w(0)$	$y(0)$	$(w_{II} \ w_{III})$	$(a_{(1)} \ b_{(2)})$
0 to less than 1	Increased demand for goods ₂	Unchanged wage $w(0)$	Moving towards $y(1)$	$(w_{II} \ w_{III})$	$(a_{(1)} \ b_{(2)})$
1	Reached production limit	Wage remains $w(0)$	Reaching $y(1)$	$(w_{II} \ w_{III})$	$(a_{(1)} \ b_{(2)})$
1 to less than 2	Increase in wages in Country 2	Wage in Country 2 moving towards $w(1)$	Unchanged $y(1)$	$(w_{II} \ w_{III})$	$(a_{(1)} \ b_{(2)})$
2	Reached minimal wage face w_{III} and expansion of maximal technology face to $(a_{(1)} \ a_{(2)} \ b_{(2)})$	Reached $w(1)$.	Unchanged $y(1)$	(w_{III})	$(a_{(1)} \ a_{(2)} \ b_{(1)})$
2 and beyond	Demand can change in any direction from $y(1)$	Remains at $w(1)$	Can change in any direction based on demand	(w_{III})	$(a_{(1)} \ a_{(2)} \ b_{(1)})$

15.8 Face Movement Part 2

Let the economy be situated on the diagonal line pointing northeast of $P_{III}^{(3)}$, denoted by $y(2)$. Wages remain at the position of $w(1)$ (Figure 21). Even if the economy attempts to move northeast of $y(2)$, it cannot move beyond the feasible region.

During this process, there is excess demand for goods, leading to competition for labor between country A and country B. As a result, wages increase, accompanied by a rise in prices (wage-price spiral). Wages move northeast along W_{III} .

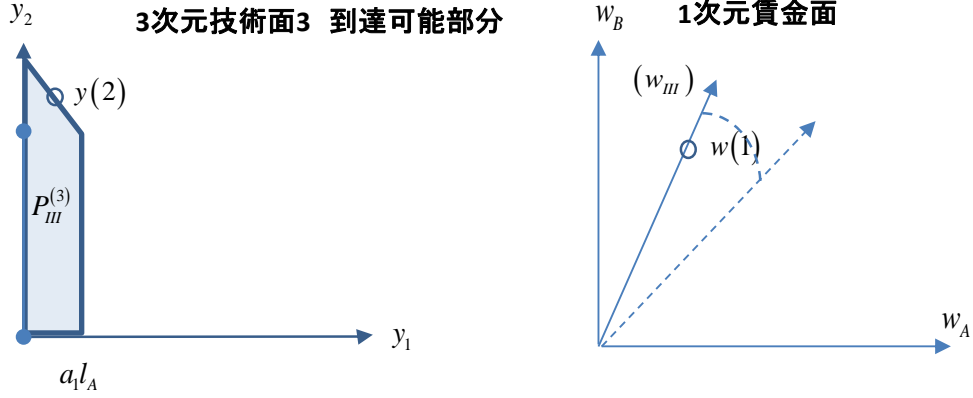


図 21 Cannot move northeast of the 3-dimensional technology face

Suppose the economy changes its demand such that it shifts southeast along the dashed line of $P_{III}^{(3)}$. During this process, wages remain constant. Let's assume the economy reaches the point

$$(a_1 \hat{l}_A, b_2 \hat{l}_B) = y(4)$$

. When there is an increase in demand for goods 1 only, only the labor market of country A experiences excess demand, resulting in a rise of w_A . Wages move from $w(1)$ to $w(5)$ (Figure). When wages move towards $w(5)$, wages enter the interior of $(W_{II} W_{III})$. As wages move to an interior point, $w(4)$, the maximal technology face shrinks to $P^{(2)}$. During this process, the output quantity remains constant at $y(4)$ (see Figure).

When wages reach $w(5)$ in the wage face W_{II} , the maximal technology face expands to $P_{II}^{(3)}$ (additional technology $b_{(1)}$, allowing country B to produce goods 1). From that point on, the economy can adjust to a decrease in output quantity of goods 2 and an increase in output quantity of goods 1 (see Figure 22).

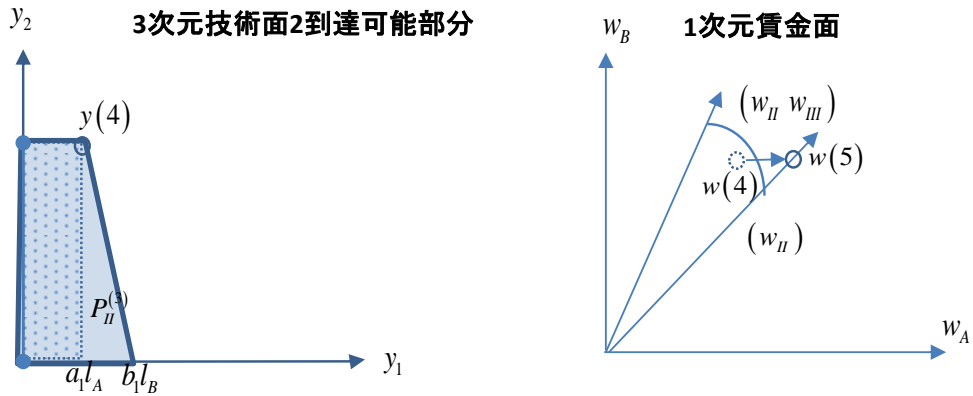


図 22 The attainable region of the commodity component expands to the 3D technology face 2

15.9 Labor Vector Embodied in Technology

15.9.1 Assumed Numeric Values

In this section, a concrete numerical example is used for illustration (see Table 26).

表 26 Assumed Numerical Values

Technology Number	1	2	3	4
	$a_{(1)}$	$a_{(2)}$	$b_{(1)}$	$b_{(2)}$
Goods1	1	0	2	0
Goods2	0	1	0	3
Labor A	-1	-1	0	0
Labor B	0	0	-1	-1
Country A has higher relative productivity in goods1 compared to goods2, with values of			a_1/b_1 0.5	a_2/b_2 0.33

15.9.2 Representation of Labor Vector Embodied in Commodities

There are two eligible facets $(a_{(1)} \ b_{(1)} \ b_{(2)})$, $(a_{(1)} \ a_{(2)} \ b_{(2)})$.

First, we discuss $(a_{(1)} \ b_{(1)} \ b_{(2)})$. The frame matrix of $(a_{(1)} \ b_{(1)} \ b_{(2)})$ is presented as follows:

The frame matrix of $(a_{(1)} \ b_{(1)} \ b_{(2)})$ is displayed in Table 27:

表 27 Frame Matrix of $(a_{(1)} \ b_{(1)} \ b_{(2)})$

Technology Number	1	3	4
Goods1	1	1	0
Goods2	0	0	1
Labor A	-1	0	0
Labor B	0	-0.5	-0.33

However, the values are normalized by the output quantity. The labor matrix embodied in commodities is described by the following table:

表 28 Labor Matrix Embodied in Commodities for $(a_{(1)} \ b_{(1)} \ b_{(2)})$

Goods Number	1	2
Country A	0.5	0
Country B	0.25	0.33

The technology labor matrix embodied is shown in the following table:

表 29 Technology Labor Matrix Embodied for $(a_{(1)} \ b_{(1)} \ b_{(2)})$

Technology Number	1	3	4
Country A	-0.5	0.5	0
Country B	0.25	-0.25	0

Next, we will explain facet $(a_{(1)} \ a_{(2)} \ b_{(2)})$.The frame matrix of $(a_{(1)} \ a_{(2)} \ b_{(2)})$ is as follows:

表 30 Frame matrix of $(a_{(1)} \ a_{(2)} \ b_{(2)})$

Technology Number	1	3	4
Goods A	1	0	0
Goods B	0	1	1
Labor A	-1	-1	0
Labor B	0	0	-0.33

Where output levels were normalized.The labor vector embodied in commodities is represented as follows:

表 31 Labor matrix embodied in commodities of $(a_{(1)} \ a_{(2)} \ b_{(2)})$

Goods Number	1	2
Country A	1	0.5
Country B	0	0.17

The technology labor matrix embodied is denoted as follows:

表 32 Technology labor matrix embodied of $(a_{(1)} \ a_{(2)} \ b_{(2)})$

Technology Number	1	3	4
Country A	0	-0.5	0.5
Country B	0	0.17	-0.17

15.9.3 Maximum relative wage by country and reference labor amount by facet for each country

表 33 Reference labor amount and relative wage on two technology facets

Facet		$(a_{(1)} \ b_{(1)} \ b_{(2)})$	$(a_{(1)} \ a_{(2)} \ b_{(2)})$
Reference labor amount	Country A	2	3
	Country B	0.5	0.33
Relative wage	Country A	0.33	0.25
	Country B	0.67	0.75

On facet $(a_{(1)} \ b_{(1)} \ b_{(2)})$, the reference labor amount for Country A is minimized, and this aligns with Country A having the maximum relative wage (equivalently, the maximum wage rate w_A/w_B) on this facet. On facet $(a_{(1)} \ a_{(2)} \ b_{(2)})$, the reference labor amount of country B is minimized, which is consistent with the fact that the relative wage of country B is maximized (equivalently, the wage rate w_A/w_B is minimized) on this facet.

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Appendix1 Derivation of 3-Dimensional Technology Face and 1-Dimensional Value Face

The normal vector of 3-Dimensional Technology Face is a solution to the following equation:

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & 0 & b_2 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} = 0, \quad (\text{Appendix1.1})$$

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} 0 & a_2 & -1 & 0 \end{bmatrix}' \leq 0. \quad (\text{Appendix1.2})$$

Expanding this equation yields the desired result.Solving the system of equations using ' q_4 ', we obtain the following relationships: the second equation implies ' $q_1 = q_4/b_1$ ', and the third equation implies ' $q_2 = q_4/b_2$ '. Substituting these into the first equation leads to ' $y_3 = a_2 y_2 = a_2/b_2 y_4$ '. While the final inequality ' $q_3 \geq q_2 a_2$ ' holds, substituting the solution into this condition yields:

$$\left(\frac{A_1}{B_1} - \frac{A_2}{B_2} \right) Q_4 \geq 0. \quad (\text{Appendix1.3})$$

Dividing by the comparative advantage condition (15.2), we have:

$$Q_4 \geq 0. \quad (\text{Appendix1.4})$$

Utilizing ' $q_4 = 1$ ' as the representative values, we set:

$$Q_1 = \frac{1}{B_1}, Q_2 = \frac{1}{B_2}, Q_3 = \frac{Q_1}{B_1}, Q_4 = 1. \quad (\text{Appendix1.5})$$

The normal 'q' of the 3-dimensional technology face3 corresponds to the solution given by the following expression:

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & b_2 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 0, \quad (\text{Appendix1.6})$$

$$\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} b_1 & 0 & 0 & -1 \end{bmatrix}' \leq 0. \quad (\text{Appendix1.7})$$

Expanding these equations:

$$a_1 q_1 - q_3 = 0, \quad (\text{Appendix1.8})$$

$$a_2 q_2 - q_3 = 0, \quad (\text{Appendix1.9})$$

$$b_2 q_2 - q_4 = 0, \quad (\text{Appendix1.10})$$

$$b_1 q_1 - q_4 \leq 0. \quad (\text{Appendix1.11})$$

Solving this system of equations with q_3 . From the first equation, $q_1 = q_3/a_1$. From the second equation, $q_2 = q_3/a_2$. Substituting the result of q_2 into the third equation, it becomes $q_4 = (b_2/a_2) q_3$. The final inequality is given by $q_4 \geq b_1 q_1$, but substituting the solution of q_1, q_4 into this condition yields

$$\left(\frac{b_2}{a_2} - \frac{b_1}{a_1} \right) q_3 \geq 0. \quad (\text{Appendix1.12})$$

Dividing by the comparative advantage condition (15.2), we have

$$q_3 \geq 0. \quad (\text{Appendix1.13})$$

Let $q_3 = 1$ represent the representative values.

$$q_1 = \frac{1}{a_1}, q_2 = \frac{1}{a_2}, q_3 = 1, q_4 = \frac{b_2}{a_2}. \quad (\text{Appendix1.14})$$

The normal of 3-dimensional technology face 4 is a solution to the following equation. Expanding the matrix equations provided in Equations , we have:

$$a_1 q_1 - q_3 = 0, \quad (\text{Appendix1.15})$$

$$a_2 q_2 - q_3 = 0, \quad (\text{Appendix1.16})$$

$$b_1 q_1 - q_4 = 0, \quad (\text{Appendix1.17})$$

$$b_2 q_2 - q_4 \leq 0. \quad (\text{Appendix1.18})$$

Solving these simultaneous equations yields the following results: From the first equation, we have $q_3 = a_1 q_1$. From the second equation, $q_3 = a_2 q_2$. Substituting the results from Equations

$q_1 = q_3/a_1$ and $q_2 = q_3/a_2$ into the third equation, we find that $b_1q_1 = b_2q_2 \leq 0$. From the last inequality, we have $q_4 \geq b_2q_2$, but substituting the solution of q_1, q_4 into this condition yields

$$\left(\frac{b_1}{a_1} - \frac{b_2}{a_2}\right)q_3 \geq 0. \quad (\text{Appendix1.19})$$

Dividing by the comparative advantage condition (15.2), we get

$$q_3 \leq 0. \quad (\text{Appendix1.20})$$

Let the representative values be denoted as $q_3 = -1$:

$$q_1 = -\frac{1}{a_1}, \quad q_2 = -\frac{1}{a_2}, \quad q_3 = -1, \quad q_4 = -\frac{b_1}{a_1}. \quad (\text{Appendix1.21})$$

Appendix2 Technical Surface With Fixed l_2

To visually illustrate the differences in technology face, we derive and graph the eligible technology face with fixed $l_2 < 0$. We denote the fixed value of l_2 as $\bar{l}_2 < 0$. For the operating scale vector $x' = (x_1, x_2, x_3, x_4)$, operating scales of the technologies not included in the technology face are set to 0.

We start with the simplest 2-dimensional technology face, denoted as $(a_{(1)} \ b_{(2)})$, which is $x_2 = x_3 = 0$.

$$Y_1 = X_1 \cdot a_1, \quad (\text{Appendix2.1})$$

$$Y_2 = X_4 \cdot b_2, \quad (\text{Appendix2.2})$$

$$L_1 = -X_1, \quad (\text{Appendix2.3})$$

$$\bar{L}_2 = -X_4. \quad (\text{Appendix2.4})$$

Solving by eliminating x_1, x_4 , we get:

$$Y_1 = -a_1 L_1, \quad (\text{Appendix2.5})$$

$$Y_2 = -b_2 \bar{L}_2. \quad (\text{Appendix2.6})$$

y_2 is a fixed value, and y_1 is a half-line. Parallel to the plane translated from $y_1 \cdot l_1$ to $y_2 = -b_2 \bar{l}_2$, there exists a half-line $y_1 = -a_1 l_1$ on that plane (see Figure 23).

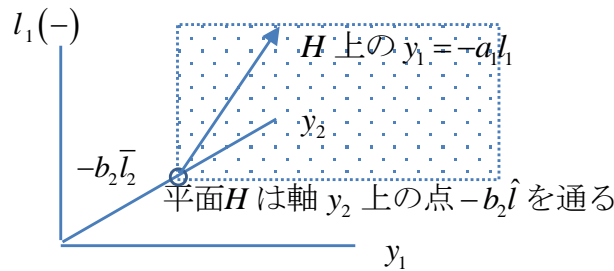


図 23 2-Dimensional technology face with l_2 fixed

Next, we calculate the $(a_{(1)} \ b_{(1)} \ b_{(2)})$ for 3-dimensional technology face 2, which is $x_2 = 0$.

$$Y_1 = X_1 A_1 + X_3 B_1, \quad (\text{Appendix2.7})$$

$$Y_2 = X_4 B_2, \quad (\text{Appendix2.8})$$

$$L_1 = -X_1, \quad (\text{Appendix2.9})$$

$$\bar{L}_2 = -X_3 - X_4. \quad (\text{Appendix2.10})$$

From the second equation, $x_4 = y_2/b_2$. Substituting this into the fourth equation, we get $x_3 = -\bar{l}_2 - x_4$. Substituting this result along with the third equation into the first equation, we have,

$$\begin{aligned} Y_1 &= -A_1 L_1 - B_1 (\bar{L}_2 + \frac{Y_2}{B_2}) \\ &= -A_1 L_1 - B_1 \bar{L}_2 - \frac{B_1}{B_2} Y_2 \Leftrightarrow \\ Y_1 + \frac{B_1}{B_2} Y_2 &= -A_1 L_1 - B_1 \bar{L}_2. \end{aligned} \quad (\text{Appendix2.11})$$

The projection onto the $y_1 \cdot y_2$ plane is represented by a downward sloping line. When l_1 algebraically decreases (its absolute value increases), the line shifts parallel upwards (see Figure 24).

Figure 24 represents the 3D technology face2. The right figure illustrates the projection onto the $y_1 \cdot y_2$ plane. For $l_1 = 0$, the plot is a right-leaning line segment close to the origin, and as l_1 increases, the slope remains constant while shifting parallel upwards. The figure on the right is a replot of the left figure in the $y_1 \cdot y_2 \cdot l_1$ coordinate system. When $l_1 = 0$, it is a downward sloping straight line segment on the lower left of the $y_1 \cdot y_2$ plane. As l_1 increases, the $y_1 \cdot y_2 \cdot l_1$ coordinate system moves upward and to the right, tracing a faint navy blue diagonal face.

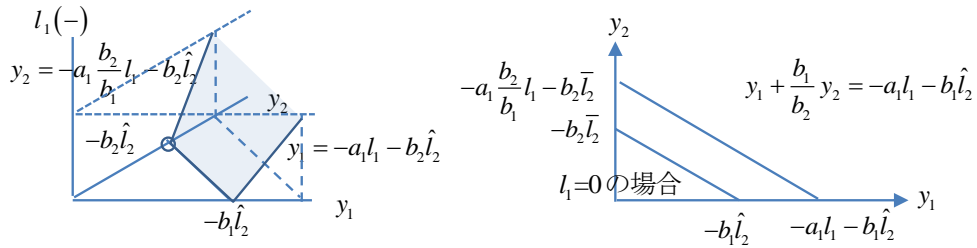


図 24 3rd dimensional technology face2 with fixed l_2

Finally, the calculation for 3rd dimensional technology face3 $(a_{(1)} \ a_{(2)} \ b_{(2)})$ is performed. It is given by $x_3 = 0$.

$$y_1 = x_1 a_1, \quad (\text{Appendix2.12})$$

$$y_2 = x_2 a_2 + x_4 b_2, \quad (\text{Appendix2.13})$$

$$l_1 = -x_1 - x_2, \quad (\text{Appendix2.14})$$

$$\bar{l}_2 = -x_4. \quad (\text{Appendix2.15})$$

From the 1st equation, $x_1 = y_1/a_1$ follows. Substituting this into the 3rd equation yields $x_2 = -l_1 - x_1 = -l_1 - y_1/a_1$. When substituting the result into the second equation, which is the fourth equation, we have

$$\begin{aligned} Y_2 &= -A_2 \left(L_1 + \frac{Y_1}{A_1} \right) - B_2 \bar{L}_2 \\ &= -A_2 L_1 - \frac{A_2}{A_1} Y_1 - B_2 \bar{L}_2 \Leftrightarrow \\ \frac{A_2}{A_1} Y_1 + Y_2 &= -A_2 L_1 - B_2 \bar{L}_2. \end{aligned} \quad (\text{Appendix2.16})$$

The projection onto the $y_1 \cdot y_2$ plane is represented by a downward-sloping straight line. As l_1 algebraically decreases (its absolute value increases), the line shifts parallelly upwards (see Figure 25).

Figure 25 depicts the 3-dimensional technology face3. The right figure shows the projection onto the $y_1 \cdot y_2$ plane. The graph at $l_1 = 0$ displays a region of downward-sloping straight lines close to the origin, and as l_1 increases, the slopes remain constant while shifting parallelly upwards. To highlight the difference from technology face2, which is shown with dashed lines.

The plot on the right is redrawn in the $y_1 \cdot y_2 \cdot l_1$ coordinate system, shown in the left figure. For $l_1 = 0$, the region on the $y_1 \cdot y_2$ plane depicts downward-sloping straight lines in the lower-left. As l_1 increases, the trajectory moves upwards to the right in the $y_1 \cdot y_2 \cdot l_1$ coordinate system, creating a faint navy slanted face.

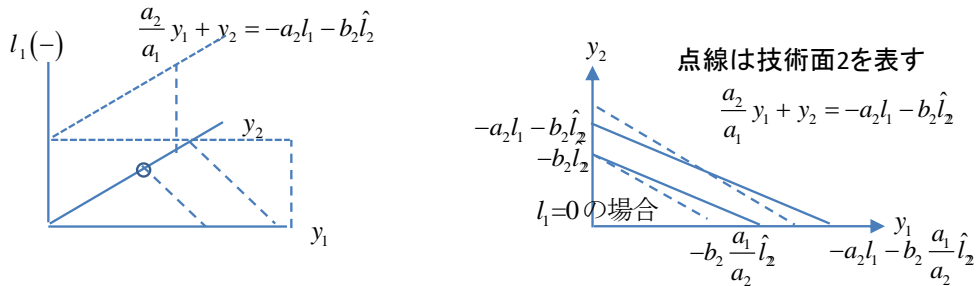


図 25 3D technology face3 with fixed l_2

Comparing 3D technology face2 with technology face3, the positions on the y_1 axis are different at the level of l_1 . For technology face3 at $l_1 = 0$, the position is given by,

$$b_2 \frac{a_1}{a_2} - b_1 = b_2 \left(\frac{a_1}{a_2} - \frac{b_1}{b_2} \right) > 0, \text{ based on the comparative advantage condition (15.2),} \\ \text{(Appendix2.17)}$$

thus the position is to the right of 3D technology face2.

Comparing technology face3 with 3D technology face2, the positions on the y_2 axis vary at the level of l_1 . The position for the case of $l_1 = 0$ is identical at $-b_2 \bar{l}_2$ for technology face2 and face3. When $l_1 > 0$,

$$A_2 - A_1 \frac{B_2}{B_1} = A_1 \left(\frac{A_2}{A_1} - \frac{B_2}{B_1} \right) < 0 \quad \text{(Appendix2.18)}$$

it is located below the three-dimensional technology face2. The last inequality is derived using the comparative advantage condition (15.2).

When $l_1 > 0$, the position on the y_1 axis of the technology face3 compared to the 3-dimensional technology face2 can be determined using the previous result:

$$\underbrace{\left(A_2 - A_1 \frac{B_2}{B_1} \right)}_{\text{Factor located to the left of technology face2 due to being negative}} (-L_1) + \underbrace{\left(B_2 \frac{A_1}{A_2} - B_1 \right)}_{\text{Factor located to the right of technology face2 due to being positive}} (-\bar{L}_2). \quad \text{(Appendix2.19)}$$

The position is determined by the magnitude of l_1 . When the absolute value of l_1 exceeds a certain level, the first term predominates, resulting in an overall negative value, positioning it to the left of technology face2. The graph overlaying 3-dimensional technology faces 2 and 3 is shown in Figure 26.

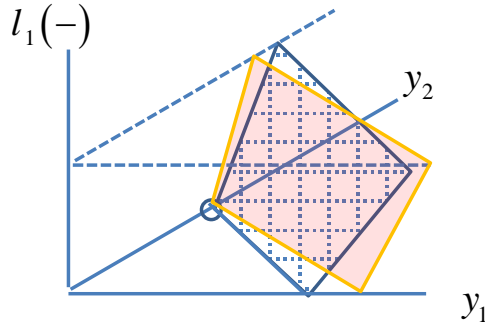


图 26 3-dimensional technology faces 2 and 3 with fixed l_2